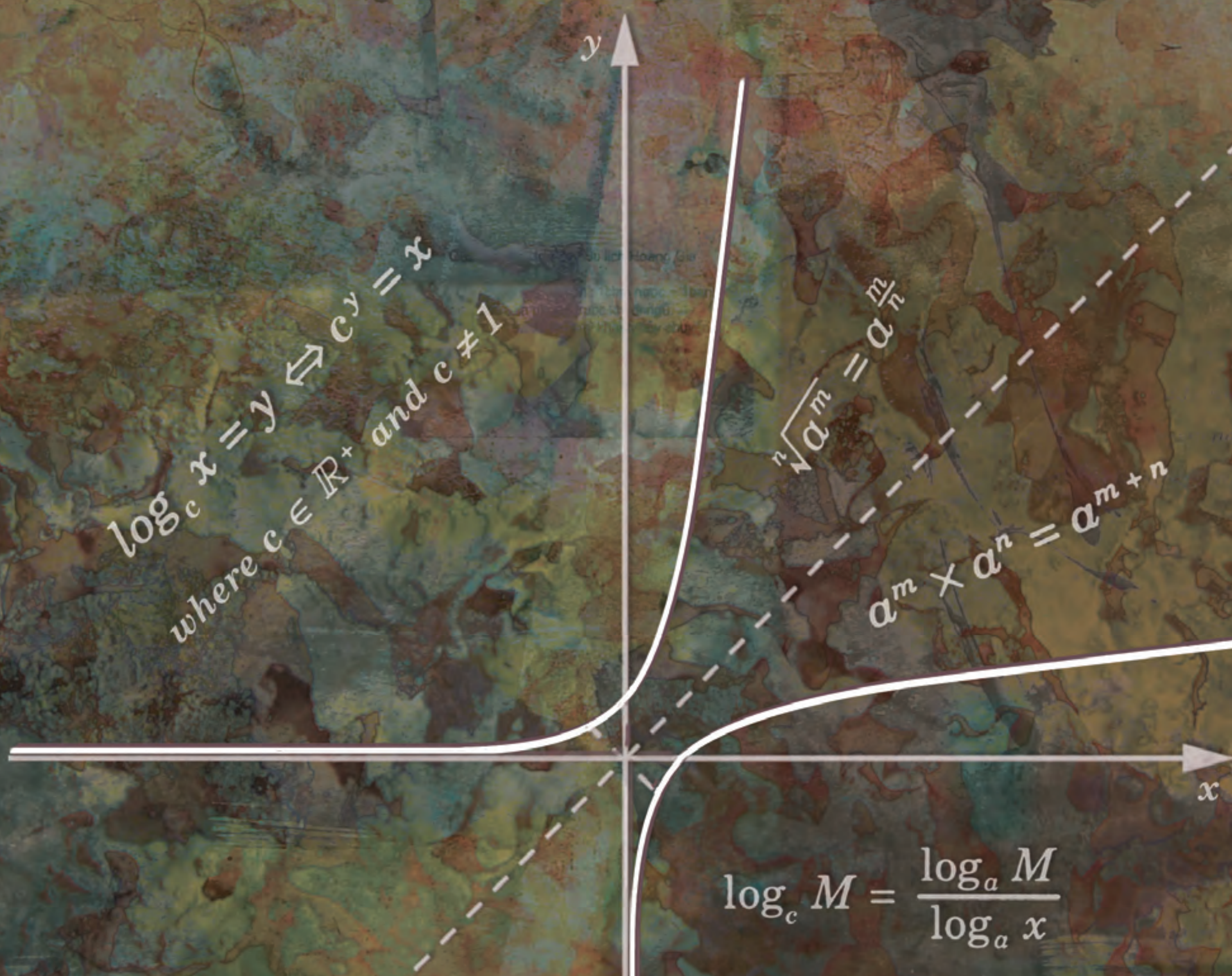
MTH-5107-2

Exponential and Logarithmic Functions and Equations



sofad

MTH-5107-2

EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND EQUATIONS

sofad

Mathematics Project Coordinator: Jean-Paul Groleau

Author: Serge Dugas

Update version: Line Régis
Catherine Paris

Content Revision: Jean-Paul Groleau
Alain Malouin

Pedagogical Revision: Jean-Paul Groleau

Translation: Claudia de Fulviis

Linguistic Revision: Johanne St-Martin

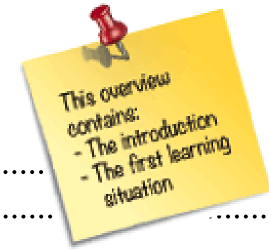
Electronic Publishing: P.P.I. inc.

Cover Page: Daniel Rémy

First Printing: 2008

Legal Deposit — 2008
Bibliothèque et Archives nationales du Québec
Bibliothèque et Archives Canada
ISBN 978-2-89493-318-3

TABLE OF CONTENTS



UNITS

INTRODUCTION TO THE PROGRAM FLOWCHART

WELCOME TO THE WORLD OF MATHEMATICS

This mathematics program has been developed for adult students enrolled either with Adult Education Services of school boards or in distance education. The learning activities have been designed for individualized learning. If you encounter difficulties, do not hesitate to consult your teacher or to telephone the resource person assigned to you. The following flowchart shows where this module fits into the overall program. It allows you to see how far you have come and how much further you still have to go to achieve your vocational objective. There are three possible paths you can take, depending on your goal.

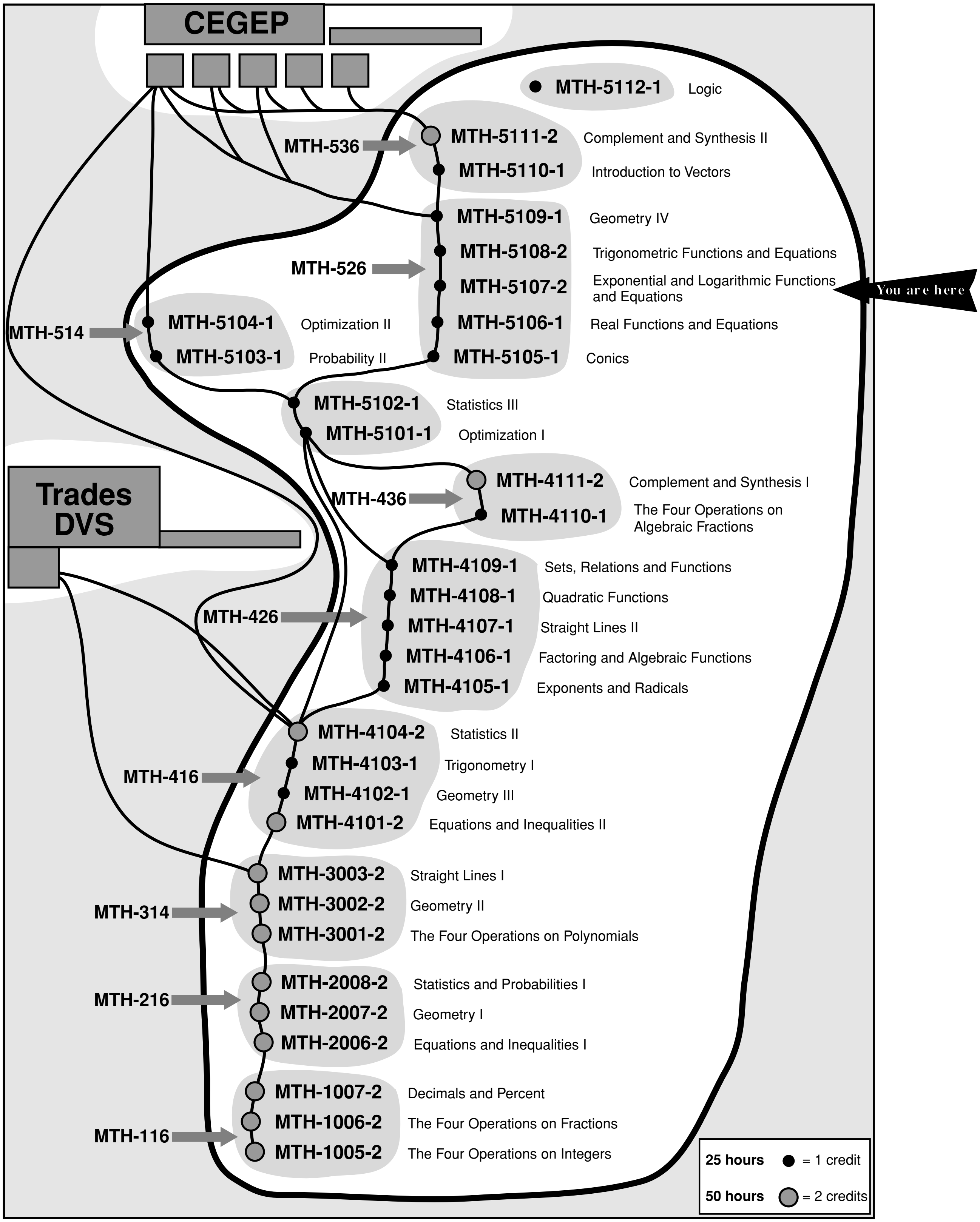
The first path, which consists of Modules MTH-3003-2 (MTH-314) and MTH-4104-2 (MTH-416), leads to a Secondary School Vocational Diploma (SSVD) and certain college-level programs for students who take MTH-4104-2.

The second path, consisting of Modules MTH-4109-1 (MTH-426), MTH-4111-2 (MTH-436) and MTH-5104-1 (MTH-514), leads to a Secondary School Diploma (SSD), which gives you access to certain CEGEP programs that do not call for a knowledge of advanced mathematics.

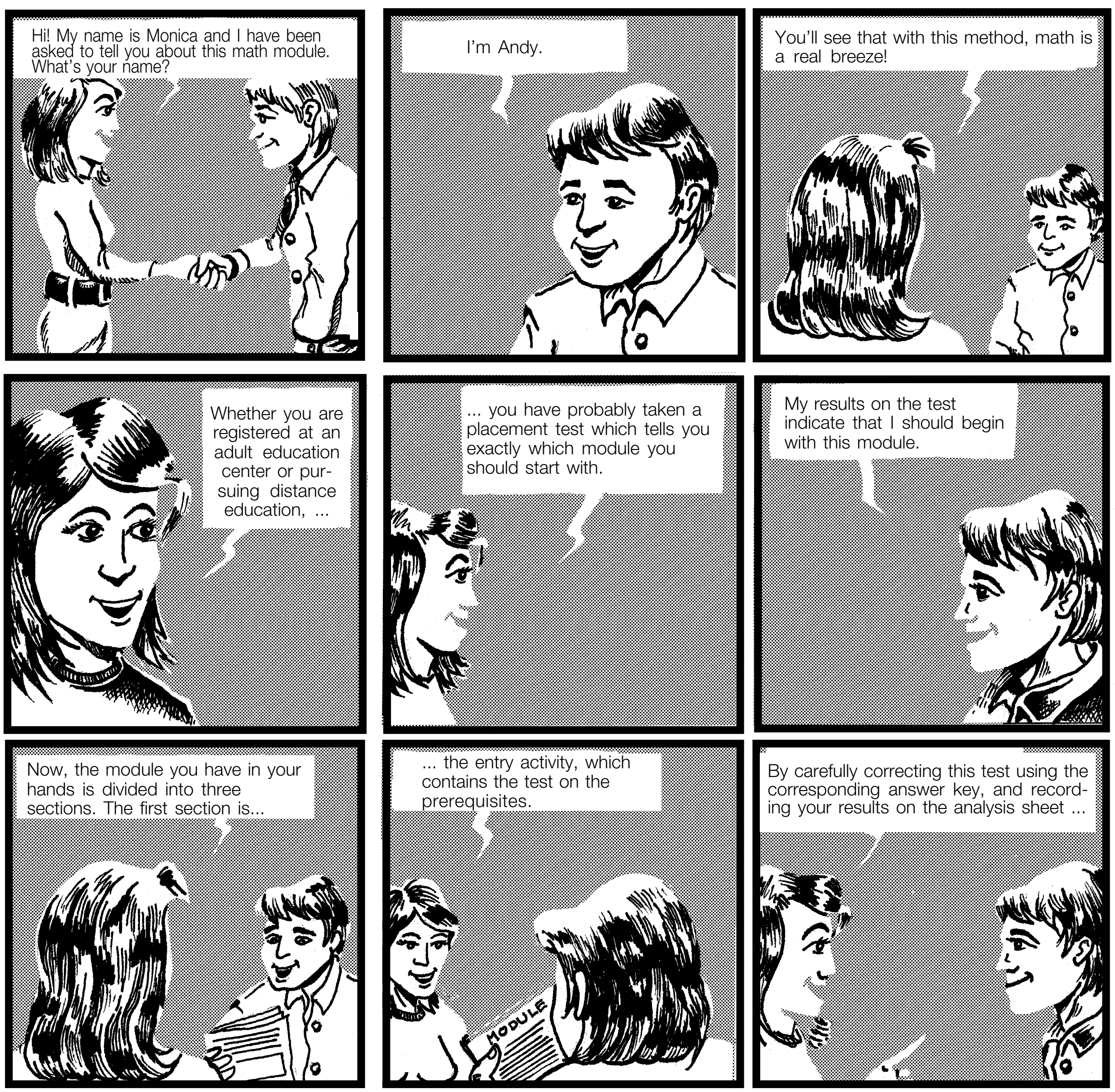
Lastly, the path consisting of Modules MTH-5109-1 (MTH-526) and MTH-5111-2 (MTH-536) will lead to CEGEP programs that require a thorough knowledge of mathematics in addition to other abilities. Good luck!

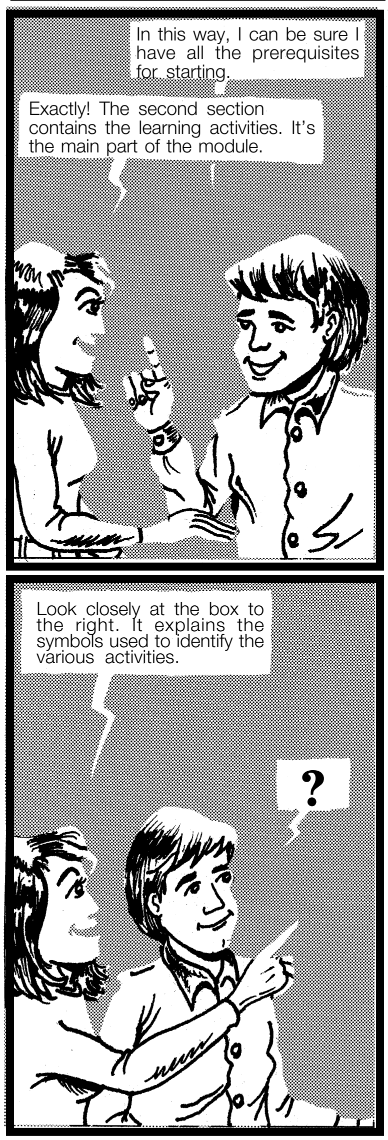
If this is your first contact with the mathematics program, consult the flowchart on the next page and then read the section "How to Use this Guide." Otherwise, go directly to the section entitled "General Introduction." Enjoy your work!

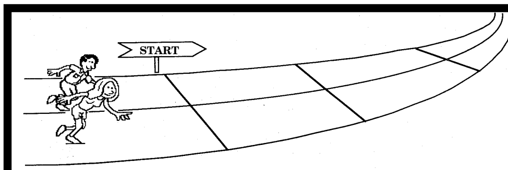
PROGRAM FLOWCHART



HOW TO USE THIS GUIDE

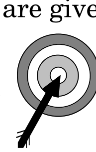




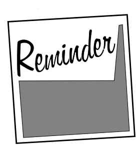


The starting line shows where the learning activities **begin**.

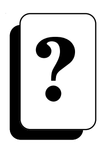
The little white question mark indicates the **questions** for which answers are given in the text.



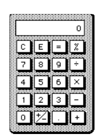
The target precedes the **objective** to be met.



The memo pad signals a **brief reminder** of concepts which you have already studied.

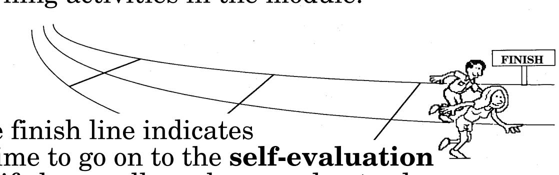


The boldface question mark indicates **practice exercises** which allow you to try out what you have just learned.



The **calculator** symbol reminds you that you will need to use your calculator.

The sheaf of wheat indicates a **review** designed to reinforce what you have just learned. A row of sheaves near the end of the module indicates the final review, which helps you to interrelate all the learning activities in the module.



Lastly, the finish line indicates that it is time to go on to the **self-evaluation test** to verify how well you have understood the learning activities.


There are also many fun things in this module. For example, when you see the drawing of a sage, it introduces a "Did you know that..."
A "Did you know that..."?
Yes, for example, short tidbits on the history of mathematics and fun puzzles. They are interesting and relieve tension at the same time.
Must I memorize what the sage says?
No, it's not part of the learning activity. It's just there to give you a breather.
It's the same for the "math whiz" pages, which are designed especially for those who love math.
They are so stimulating that even if you don't have to do them, you'll still want to.
And the whole module has been arranged to make learning easier.
For example. words in boldface italics appear in the glossary at the end of the module...
Great!
... statements in boxes are important points to remember, like definitions, formulas and rules. I'm telling you, the format makes everything much easier.
The third section contains the final review, which interrelates the different parts of the module.
There is also a self-evaluation test and answer key. They tell you if you're ready for the final evaluation.
Thanks, Monica, you've been a big help.
I'm glad! Now, I've got to run. See you!
Later ...
This is great! I never thought that I would like mathematics as much as this!

GENERAL INTRODUCTION

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In your previous courses, you learned the concepts of relations and functions. You also learned to graph a certain number of functions (linear, quadratic, absolute-integer and square-root functions), and in so doing you were able to identify their characteristics.

In this course, you will discover two new, closely related functions: the exponential function and the logarithmic function. You are probably already familiar with the term "exponential," which derives from the word "exponent." In this course, you will learn that the logarithmic function is the inverse of the exponential function.

In addition to graphing both these functions and identifying their individual characteristics, we will look at the laws governing exponential and logarithmic calculation, and you will learn to calculate the logarithm of a number using a calculator.

Lastly, before we tackle problems involving exponential and logarithmic functions, you will learn to solve exponential and logarithmic equations.

INTERMEDIATE AND TERMINAL OBJECTIVES OF THE MODULE

Module MTH-5107-2 consists of nine units and requires 50 hours of study distributed as shown below. The terminal objectives appear in boldface.

Objectives	**Number of hours***	**% (Evaluation)**
1	6	10%
2	4	10%
3	4	10%
4	6	10%
5	4	10%
6	6	20%
7	8	10%
8	4	10%
9	6	20%

* Two hours are allotted to the final evaluation.

1. Graphing an exponential function

Graph an exponential function of the form $f(x) = ac^{b(x-h)} + k$, and determine the characteristics of the function and the connections between the change in a parameter of the rule and the transformation of the corresponding Cartesian graph.

2. Finding the rule of an exponential function

Find the rule of an exponential function of the form $f(x) = \pm c^x + k$ given relevant information or its graph. The relevant information or the graph must include the equation of the asymptote or the coordinates of a point whose x-coordinate is not 0.

3. Converting an expression from exponential form to logarithmic form, and vice versa

Convert an expression from exponential form to logarithmic form and convert an expression from logarithmic form to exponential form. The exponential expressions are of the form $y = c^x$ and the logarithmic expressions are of the form $y = \log_c x$.

4. Graphing a logarithmic function

Graph a logarithmic function of the form $f(x) = a\log_c b(x - h) + k$, identify the asymptote on the graph and determine whether the function is an increasing or a decreasing function. Find the domain and range of this function and express them in set-builder notation or in interval form. Determine the connections between the change in a parameter of the rule and the transformation of the corresponding Cartesian graph.

5. Finding the rule of a logarithmic function

Find the rule of a logarithmic function of the form $f(x) = \log_c \pm(x - h)$ given relevant information or its graph. The relevant information or the graph must include the equation of the asymptote or the coordinates of a point whose x-coordinate is not 0.

6. Finding the inverse of an exponential function and a logarithmic function

Find the inverse of an exponential function of the form $f(x) = \pm c^x + k$ or the inverse of a logarithmic function of the form $f(x) = \log_c \pm(x - h)$. The inverse must be written in set-builder notation.

7. Applying the laws of logarithmic calculation

Simplify a logarithmic expression by applying the properties of logarithms. Convert an expression from exponential form to logarithmic form or, conversely, convert an expression from logarithmic form to exponential form. The exponential expressions are of the form $y = c^x$ and the logarithmic expressions are of the form $y = \log_c x$. Calculate the value of a logarithm to base 10, to base e, or to any other base. Prove and apply the properties of the following logarithms:

- $\log_c 1 = 0$
- $\log_c c = 1$
- $\log_c c^n = n$
- $\log_c M^n = n \bullet \log_c M$
- $\log_{\frac{1}{c}} M = -\log_c M$
- $\log_c M = \dfrac{\log_a M}{\log_a c}$
- $\log_c (M \bullet N) = \log_c M + \log_c N$
- $\log_c \left(\dfrac{M}{N}\right) = \log_c M - \log_c N$

where $M, N, c \in R$ and $c \neq 1$.

The given expression must not include more than three numerical or algebraic terms: each term is a logarithm or a number to be expressed in logarithmic form. Simplifying the expression may require the simple factorization of polynomials. The result is a logarithmic expression of lowest form or a numerical value.

8. Solving exponential or logarithmic equations

Solve an exponential equation in which both members can be expressed in the same base. Solve an exponential equation where the two members of the equation are not powers of the same base. The exponents of the bases are numbers or algebraic expressions of degree 1. Solve a logarithmic equation where the two members of the equation can be simplified to an expression containing only one logarithm, by using the properties of logarithms. The given equation should consist of no more than three terms. One of the members of the equation may contain an algebraic expression of degree 2.

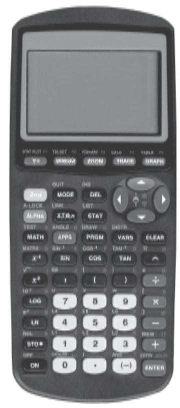
9. Solving problems involving exponential and logarithmic functions

Solve problems that involve applying concepts related to exponential or logarithmic functions. The solution may require finding the rule, drawing the graph, determining certain characteristics of the function and deducing certain information depending on the context. The rule is given if the function is a logarithmic function.

DIAGNOSTIC TEST ON THE PREREQUISTES

Instructions

1. Answer as many questions as you can.
2. You may use a calculator.



3. You may use a ruler graduated in centimetres and millimetres.
4. Write your answers on the test paper.
5. Don't waste any time. If you cannot answer a question, go on to the next one immediately.
6. When you have answered as many questions as you can, correct your answers using the answer key which follows the diagnostic test.
7. To be considered correct, your answers must be identical to those in the key. In addition, the various steps in your solution should be equivalent to those shown in the answer key.
8. Transcribe your results onto the chart which follows the answer key. This chart gives an analysis of the diagnostic test results.
9. Do the review activities that apply to each of your incorrect answers.
10. If all your answers are correct, you may begin working on this module.

1. Relation $\mathcal{R}$ is defined by the propositional form "... *is smaller than or equal to* ... " in a Cartesian product A × B, where A = {1, 2, 6} and B = {0, 2, 7}.

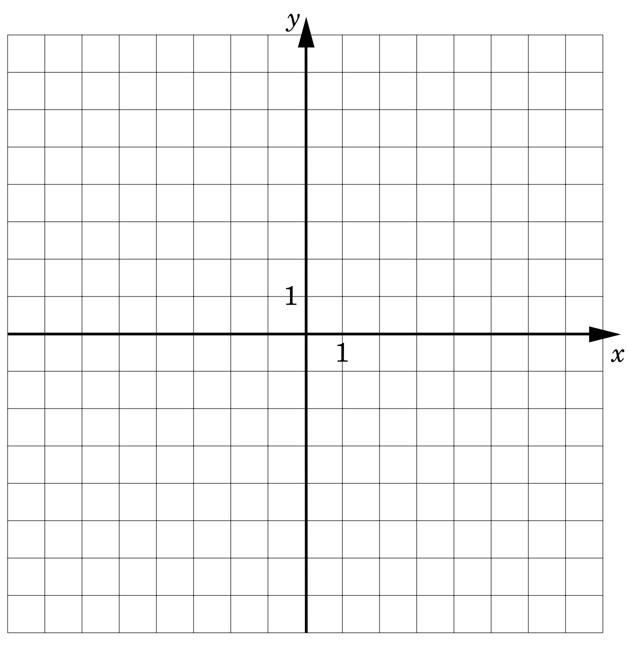
   a) Give the ordered pairs of A × B. ....

   b) Describe $\mathcal{R}$ by listing its elements. ....

   c) Describe dom $\mathcal{R}$. ....

   d) Describe ran $\mathcal{R}$. ....

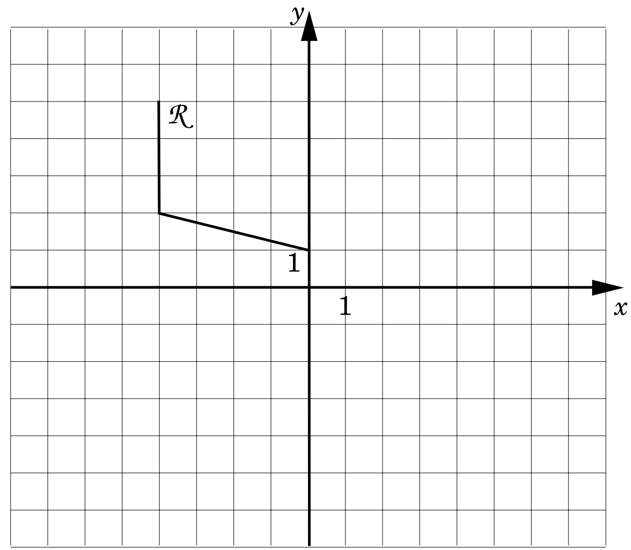
   e) Graph this relation on the grid below.



2. Find the inverse relations of the relations shown below.

   a) $\mathcal{R}$ = {(1, 4), (3, 7), (6, 9)} ....

b)



3. Let a function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = -2x + 3$.

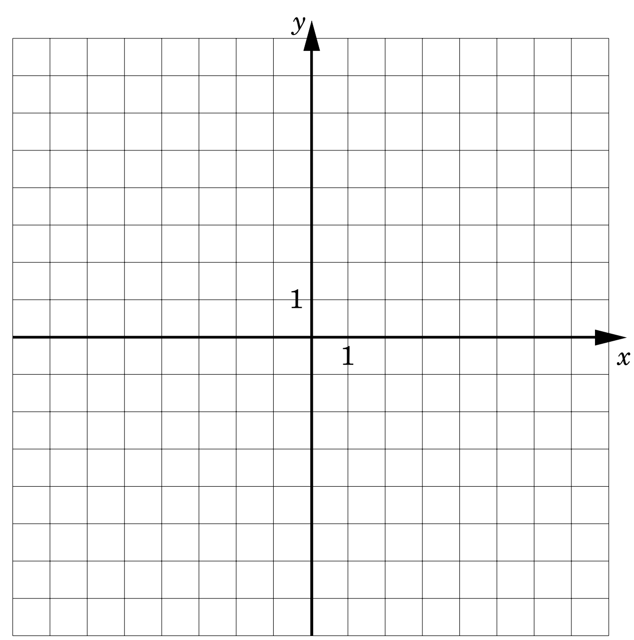
   a) What is the name of this function? ....

   b) What is its source set? ....

   c) What is its target set? ....

   d) Calculate $f(4)$. ....

   e) Draw function f on the following grid.



4. Let A = {1, 4, 9, 16}, B = {0, 1, 2, 3} and a function g: A → B be defined by the ordered pairs {(1, 1), (4, 2), (9, 3)}.

   a) What is the source set of g? ....

   b) What is its target set? ....

   c) Find dom g. ....

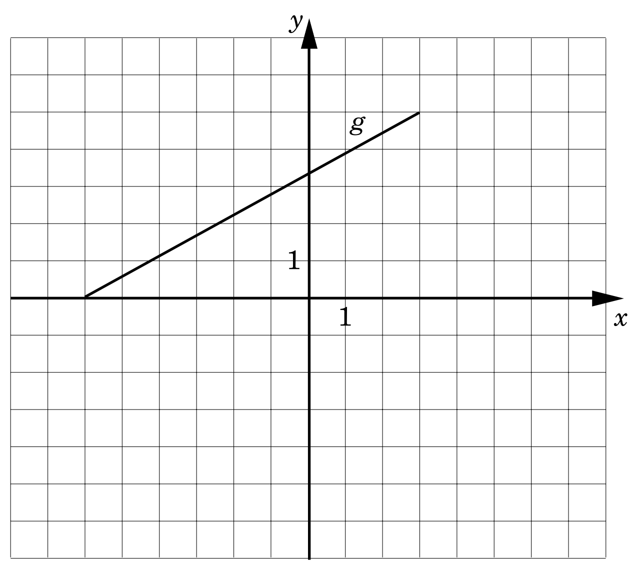
   d) Find ran g. ....

5. Find the inverse functions of the following.

   a) f: A → B, f = {(1, 2), (2, 3), (3, 4), (4, 5)}

   ....

   b)



   c) $h: \mathbb{R} \rightarrow \mathbb{R}$, $h(x) = 5x - 4$

   ....

   ....

   ....

....

....

....

6. Apply the appropriate laws of exponents in order to simplify the following exponential expressions. Use positive exponents.

a) $9^3 \times 3^2$

....

b) $(4^5 \times 32^{-2} \times 3)^2$

....

....

c) $\dfrac{8^2}{4^5 \times 2^{-2}}$

....

....

d) $\left(\dfrac{2^4 \times 27^2}{8^{-2}x^2}\right)^{-3}$

....

....

....

....

e) $\left(\dfrac{3^2a^4b \times 5^{-2}ca^3}{9^2c^{-3}b^4 \times 25^3a^{-2}}\right)^{-\frac{2}{3}}$

....

....

....

....

....

....

7. Convert the following expressions to an exponential form with a positive exponent.

a) $\sqrt[3]{125^2}$ = ...

b) $36^{-3}\sqrt[3]{6^9}$ = ...

c) $9^3\sqrt{\left(\frac{1}{27}\right)^{-4}}$ = ...

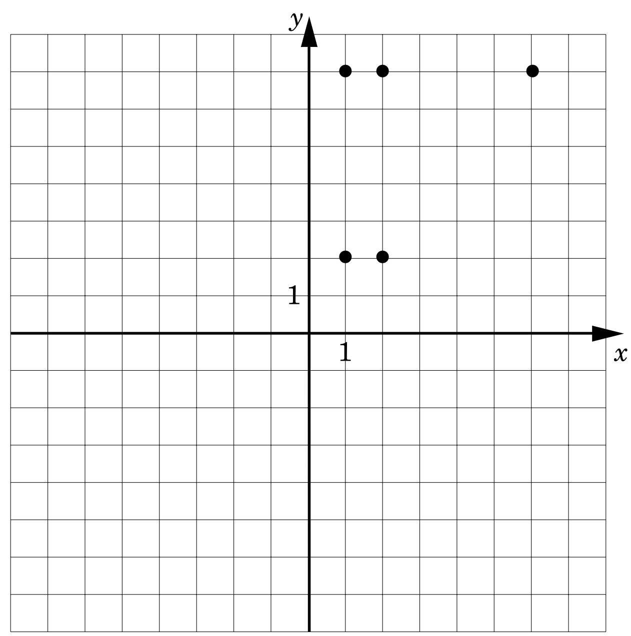
ANSWER KEY FOR THE DIAGNOSTIC TEST ON THE PREREQUISITES

1. a) $A \times B = \{(1, 0), (1, 2), (1, 7), (2, 0), (2, 2), (2, 7), (6, 0), (6, 2), (6, 7)\}$

   b) $\mathcal{R} = \{(1, 2), (1, 7), (2, 2), (2, 7), (6, 7)\}$

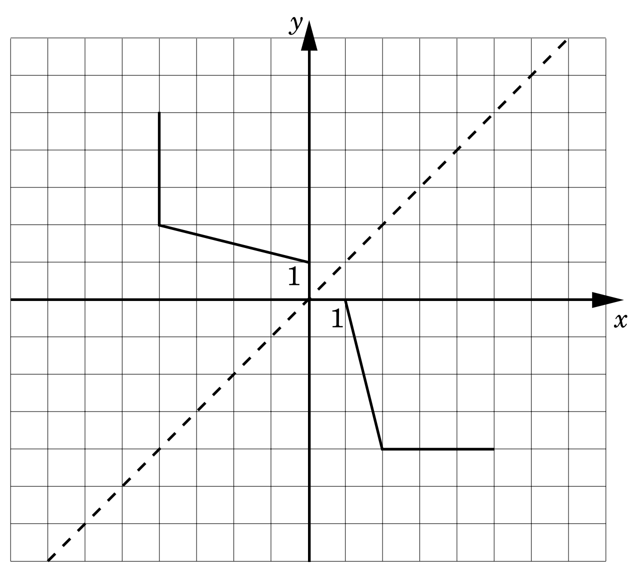
   c) Dom $\mathcal{R} = \{1, 2, 6\}$ d) Ran $\mathcal{R} = \{2, 7\}$

   e)



2. a) $\mathcal{R}^{-1} = \{(4, 1), (7, 3), (9, 6)\}$

   b)



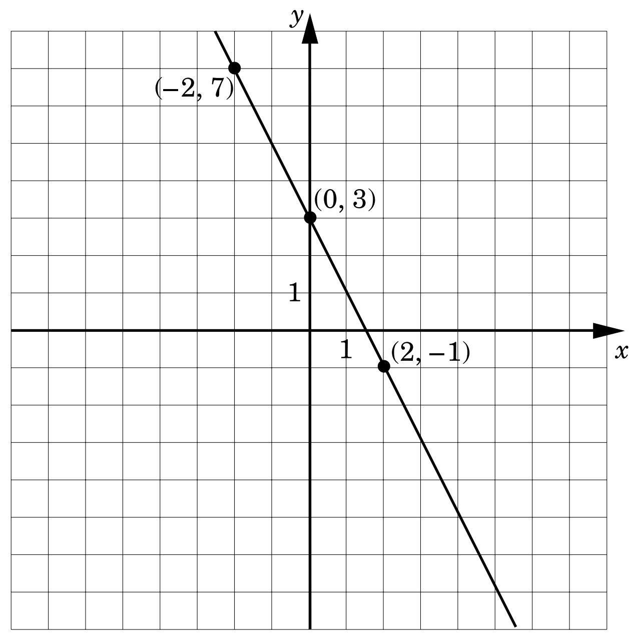
3. a) Function f b) $\mathbb{R}$ c) $\mathbb{R}$

   d) $f(4) = -2(4) + 3 = -8 + 3 = -5$

   e) $y = -2x + 3$

   - $x = -2$
     $y = -2(-2) + 3 = 4 + 3 = 7$
   - $x = 0$
     $y = -2(0) + 3 = 0 + 3 = 3$
   - $x = 2$
     $y = -2(2) + 3 = -4 + 3 = -1$

x	-2	0	2
y	7	3	-1

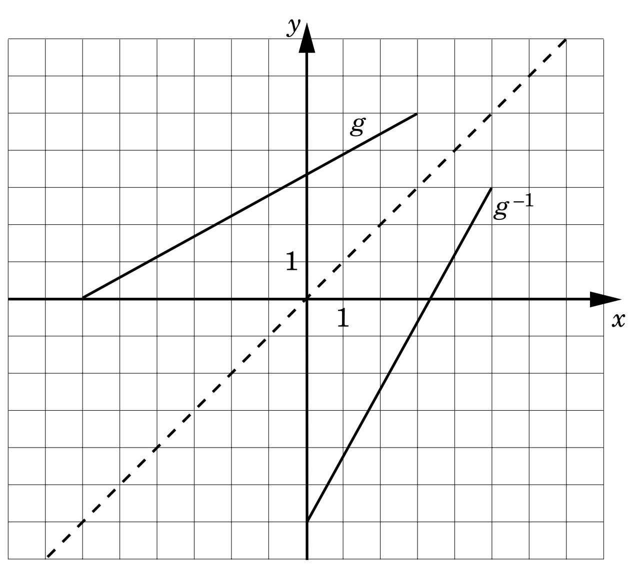


4. a) A = {1, 4, 9, 16} b) B = {0, 1, 2, 3}

   c) Dom g = {1, 4, 9} d) Ran g = {1, 2, 3}

5. a) f^{-1}: B → A, f^{-1} = {(2, 1), (3, 2), (4, 3), (5, 4)}

   b)



c) $h: \mathbb{R} \to \mathbb{R},\ h(x) = 5x - 4$

$$\begin{aligned} h(x) &= 5x - 4 \\ y &= 5x - 4 \\ x &= 5y - 4 \\ -5y &= -x - 4 \\ 5y &= x + 4 \\ y &= \frac{x+4}{5} \\ h^{-1}(x) &= \frac{x+4}{5} \end{aligned}$$

6. a) $9^3 \times 3^2 = (3^2)^3 \times 3^2 = 3^6 \times 3^2 = 3^8$

b) $(4^5 \times 32^{-2} \times 3)^2 = ((2^2)^5 \times (2^5)^{-2} \times 3)^2 = (2^{10} \times 2^{-10} \times 3)^2 =$

$(2^{10-10} \times 3)^2 = (2^0 \times 3)^2 = (1 \times 3)^2 = 3^2$

c) $$\frac{8^2}{4^5 \times 2^{-2}} = \frac{(2^3)^2}{(2^2)^5 \times 2^{-2}} = \frac{2^6}{2^{10} \times 2^{-2}} = \frac{2^6}{2^8} = 2^{6-8} = 2^{-2} = \frac{1}{2^2}$$

d) $$\left(\frac{2^4 \times 27^2}{8^{-2}x^2}\right)^{-3} = \left(\frac{2^4 \times (3^3)^2}{(2^3)^{-2}x^2}\right)^{-3} = \left(\frac{2^4 \times 3^6}{2^{-6}x^2}\right)^{-3} = \left(\frac{2^{10} \times 3^6}{x^2}\right)^{-3} =$$

$$\left(\frac{x^2}{2^{10} \times 3^6}\right)^3 = \frac{x^6}{2^{30} \times 3^{18}}$$

e) $$\left(\frac{3^2a^4b \times 5^{-2}ca^3}{9^2c^{-3}b^4 \times 25^3a^{-2}}\right)^{-\frac{2}{3}} = \left(\frac{3^2 \times 5^{-2} \times a^9c^4}{(3^2)^2 \times (5^2)^3 \times b^3}\right)^{-\frac{2}{3}} = \left(\frac{3^2 \times 5^{-2} \times a^9c^4}{3^4 \times 5^6 \times b^3}\right)^{-\frac{2}{3}} =$$

$$\left(\frac{a^9c^4}{3^2 \times 5^8 \times b^3}\right)^{-\frac{2}{3}} = \left(\frac{3^2 \times 5^8 \times b^3}{a^9c^4}\right)^{\frac{2}{3}} = \frac{3^{\frac{4}{3}} \times 5^{\frac{16}{3}} \times b^2}{a^6c^{\frac{8}{3}}}$$

7\. a) $\sqrt[3]{125^2} = \sqrt[3]{(5^3)^2} = \sqrt[3]{5^6} = 5^{\frac{6}{3}} = 5^2$

b) $36^{-3}\sqrt[3]{6^9} = (6^2)^{-3} \times 6^{\frac{9}{3}} = 6^{-6} \times 6^3 = 6^{-3} = \dfrac{1}{6^3}$ or $\dfrac{1}{2^3 \times 3^3}$

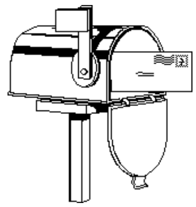
c) $9^3\sqrt{\left(\dfrac{1}{27}\right)^{-4}} = (3^2)^3\sqrt{(3^3)^4} = 3^6\sqrt{3^{12}} = 3^6 \times 3^6 = 3^{12}$

ANALYSIS OF THE DIAGNOSTIC TEST RESULTS

Questions	Answers		Review		Before going on to
	Correct	Incorrect	Section	Page	
1. a)			10.1	10.4	Units 1 to 9
b)			10.1	10.4	Units 1 to 9
c)			10.1	10.4	Units 1 to 9
d)			10.1	10.4	Units 1 to 9
e)			10.1	10.4	Units 1 to 9
2. a)			10.1	10.4	Units 1 to 9
b)			10.1	10.4	Units 1 to 9
3. a)			10.2	10.17	Units 1 to 9
b)			10.2	10.17	Units 1 to 9
c)			10.2	10.17	Units 1 to 9
d)			10.2	10.17	Units 1 to 9
e)			10.2	10.17	Units 1 to 9
4. a)			10.2	10.17	Units 1 to 9
b)			10.2	10.17	Units 1 to 9
c)			10.2	10.17	Units 1 to 9
d)			10.2	10.17	Units 1 to 9
5. a)			10.2	10.17	Units 1 to 9
b)			10.2	10.17	Units 1 to 9
c)			10.2	10.17	Units 1 to 9
6. a)			10.3	10.34	Units 1 to 9
b)			10.3	10.34	Units 1 to 9
c)			10.3	10.34	Units 1 to 9
d)			10.3	10.34	Units 1 to 9
e)			10.3	10.34	Units 1 to 9
7. a)			10.3	10.34	Units 1 to 9
b)			10.3	10.34	Units 1 to 9
c)			10.3	10.34	Units 1 to 9

- If all your answers are **correct**, you may begin working on this module.

- For each **incorrect** answer, find the related section listed in the **Review** column. Do the review activities for that section before beginning the units listed in the right-hand column under the heading **Before going on to**.



INFORMATION FOR DISTANCE EDUCATION STUDENTS

You now have the learning material for MTH-5107-2 and the relevant homework assignments. Enclosed with this package is a letter of introduction from your tutor, indicating the various ways in which you can communicate with him or her (e.g. by letter or telephone), as well as the times when he or she is available. Your tutor will correct your work and help you with your studies. Do not hesitate to make use of his or her services if you have any questions.

DEVELOPING EFFECTIVE STUDY HABITS

Learning by correspondence is a process which offers considerable flexibility, but which also requires active involvement on your part. It demands regular study and sustained effort. Efficient study habits will simplify your task. To ensure effective and continuous progress in your studies, it is strongly recommended that you:

- draw up a study timetable that takes your work habits into account and is compatible with your leisure and other activities;

- develop a habit of regular and concentrated study.

The following guidelines concerning theory, examples, exercises and assignments are designed to help you succeed in this mathematics course.

Theory

To make sure you grasp the theoretical concepts thoroughly:

1. Read the lesson carefully and underline the important points.

2. Memorize the definitions, formulas and procedures used to solve a given problem; this will make the lesson much easier to understand.

3. At the end of the assignment, make a note of any points that you do not understand using the sheets provided for this purpose. Your tutor will then be able to give you pertinent explanations.

4. Try to continue studying even if you run into a problem. However, if a major difficulty hinders your progress, contact your tutor before handing in your assignment, using the procedures outlined in the letter of introduction.

Examples

The examples given throughout the course are applications of the theory you are studying. They illustrate the steps involved in doing the exercises. Carefully study the solutions given in the examples and redo the examples yourself before starting the exercises.

Exercises

The exercises in each unit are generally modeled on the examples provided. Here are a few suggestions to help you complete these exercises.

1. Write up your solutions, using the examples in the unit as models. It is important not to refer to the answer key found on the coloured pages at the back of the module until you have completed the exercises.

2. Compare your solutions with those in the answer key only after having done all the exercises. **Careful!** Examine the steps in your solutions carefully, even if your answers are correct.

3. If you find a mistake in your answer or solution, review the concepts that you did not understand, as well as the pertinent examples. Then redo the exercise.

4. Make sure you have successfully completed all the exercises in a unit before moving on to the next one.

Homework Assignments

Module MTH-5107-2 comprises three homework assignments. The first page of each assignment indicates the units to which the questions refer. The assignments are designed to evaluate how well you have understood the material studied. They also provide a means of communicating with your tutor.

When you have understood the material and have successfully completed the pertinent exercises, do the corresponding assignment right away. Here are a few suggestions:

1. Do a rough draft first, and then, if necessary, revise your solutions before writing out a clean copy of your answer.

2. Copy out your final answers or solutions in the blank spaces of the document to be sent to your tutor. It is best to use a pencil.

3. Include a clear and detailed solution with the answer if the problem involves several steps.

4. Mail only one homework assignment at a time. After correcting the assignment, your tutor will return it to you.

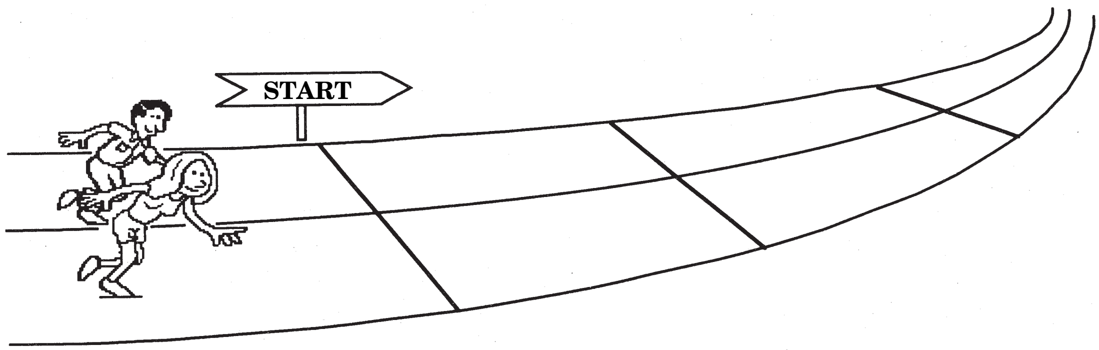
In the section "Student's Questions", write any questions which you wish to have answered by your tutor. He or she will give you advice and guide you in your studies, if necessary.

In this course

Homework Assignment 1 is based on units 1 to 5.
Homework Assignment 2 is based on units 6 to 9.
Homework Assignment 3 is based on units 1 to 9.

CERTIFICATION

When you have completed all your work, and provided you have maintained an average of at least 60%, you will be eligible to write the examination for this course.



UNIT 1

GRAPHING EXPONENTIAL FUNCTIONS

1.1 SETTING THE CONTEXT

The Power of Bonds

In a recent issue of Quebec savings bonds, the government offered an annual compound return of 7%. Monica, who worked all last summer, has \$2 000 to invest, and wonders how much her investment will yield if she sells her bonds in 3, 10, 15 or even 20 years. Equipped with a calculator and a piece of paper, she starts her calculations as follows:

- Initial capital $2 000.00
- After 1 year:
  $2 000 + $2 000 × 0.07 = $2 000 + $140 = $2 140.00
- After 2 years:
  $2 140 + $2 140 × 0.07 = $2 140 + $149.80 = $2 289.80
- After 3 years:
  $2 289.80 + $2 289.80 × 0.07 = $2 289.80 + $160.29 = $2 450.09

Monica soon realizes that this method will take a long time, and if she makes even the slightest mistake she will have to go back over the entire process! She wonders if there is an easier way of obtaining a quick answer to her question.

To achieve the objective of this unit, you should be able to graph an *exponential function*, determine its characteristics and find the connections between the change in a parameter and the transformation of the corresponding Cartesian graph.

You might suspect that such a formula exists. Business people, accountants and bankers surely do not manipulate mountains of figures every day just to establish the returns on their clients' investments.

Monica goes to see the manager of her local bank, Mr. Jasmin, who is delighted to give her the formula she wants!

$$f(x) = 2\,000 \times 1.07^x$$

"But it's a ***function***!", says Monica, surprised. She remembers her courses on functions.

Reminder

A function is a ***relation*** *between elements of a source set and at most one element of the target set.*

"The function I have just given you, Monica, is known as an **exponential function** because the variable x, which in this case represents the number of years, is an exponent. The amount of money you will accumulate, $f(x)$, is a function of the number of years x during which you keep your bonds. Apply my formula and compare it with your calculation sheet - you'll see!"

To calculate the accuracy of Mr. Jasmin's formula, Monica redoes the calculations for the first three years, replacing x successively by 1, 2 and 3.

$$f(x) = \$2\,000 \times 1.07^{x}$$
$$f(1) = \$2\,000 \times 1.07^{1} = \$2\,140.00$$
$$f(2) = \$2\,000 \times 1.07^{2} = \$2\,289.80$$
$$f(3) = \$2\,000 \times 1.07^{3} = \$2\,450.09$$

"Fantastic!" said Monica. "Now I can see how much money I will have in 20 years' time simply by replacing x with 20, instead of having to do a tedious calculation!"

$$f(20) = \$2\,000 \times 1.07^{20} = \$7\,739.37$$

No doubt our friend Monica will want to know more about exponential functions!

If f is a function of the following form:

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto c^x$$

Then $f(x) = c^x$ is known as an **exponential function to *base*** c and to the power of x, where c is a positive fraction or a positive integer and $c \neq 1$. The exponential function can also be written in the form $f(x) = \exp_c x$.

We can also write the function in ***set-builder notation***, i.e. $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = c^x\}$, which reads as follows: A is the set of ordered pairs (x, y) belonging to the Cartesian product of $\mathbb{R}$ times $\mathbb{R}$ such that $y = c^x$.

Example 1

$f(x) = 2^x$ is an exponential function with 2 as its base.
$f(x) = \left(\frac{1}{2}\right)^x$ is an exponential function with $\frac{1}{2}$ as its base.
$f(x) = \exp_4 x$ is an exponential function with 4 as its base.
$f(x) = 3^x$ is equivalent to $f(x) = \exp_3 x$.

Observations

1. The exponential function is a real function, since the source and target sets are in $\mathbb{R}$.

2. $c \neq 1$. If $c = 1$, then $f(x) = 1^x = 1$, regardless of the value of x. Thus, $f(x) = 1^x$ is the constant function $f(x) = 1$.

3. $c > 0$. The exponential function is defined only when the base is positive. For example, we cannot write $f(x) = (-2)^{\frac{1}{2}}$, because $(-2)^{\frac{1}{2}} = \sqrt{-2} \notin \mathbb{R}$.

Let us now see how exponential functions can be graphed.

Example 2

Graph the function defined by $f(x) = 2^x$ or $f(x) = \exp_2 x$. To do this, we assign different values to x, and find the corresponding values for $f(x)$, rounded off to the nearest thousandth.

$$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$$

$$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2} = 0.5$$

$$f(0) = 2^0 = 1$$

$$f(1) = 2^1 = 2$$

$$f(2) = 2^2 = 4$$

$$f(3) = 2^3 = 8$$

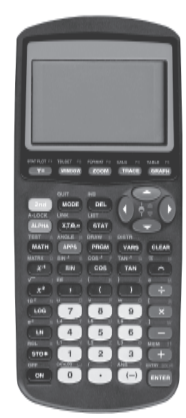
$$f(4) = 2^4 = 16$$

The calculator makes it easy to find the value of a number raised to a power (given an ***exponent****). For example, to calculate 2^{-3}, we press the following sequence of keys.*

2 y^x 3 +/− =

OR

If you are using a graphing calculator, proceed as follows.



Y =

CLEAR as many times as necessary.

2 ^ X, T, Ø, *n*

2nd TABLE (GRAPH) for the table.

This gives us the following table of values.

x	-2	-1	0	1	2	3	4
$f(x)$	0.25	0.5	1	2	4	8	16

Each of the ordered pairs is then represented in the Cartesian graph by a point. We can replace x by each of the elements in the set of real numbers, and hence find an infinite number of ordered pairs. By joining the points, we obtain an uninterrupted curve. The more ordered pairs we select, the more precise the graph will be.

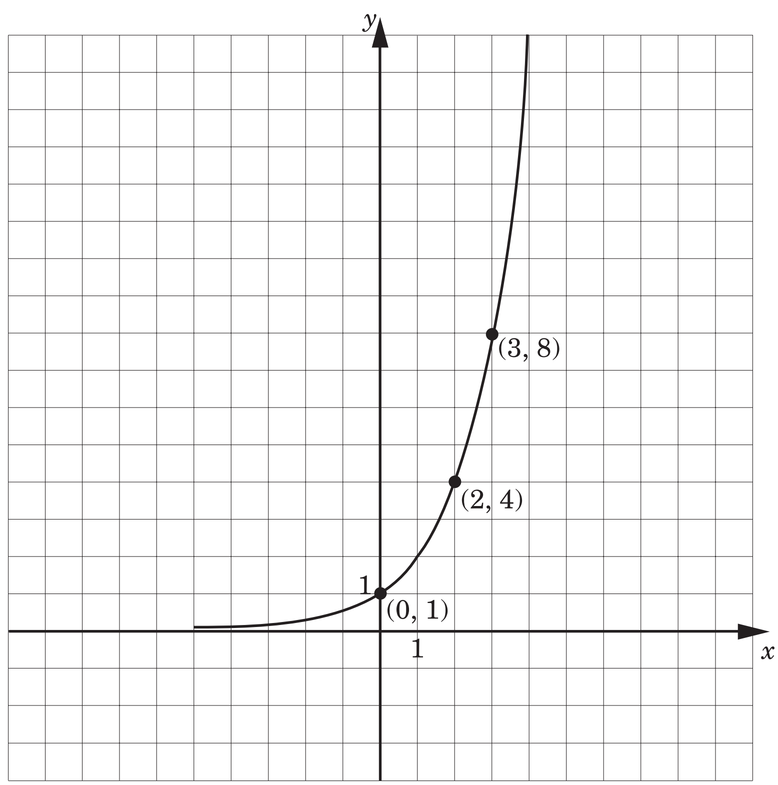


Fig. 1.1 Graph of $f(x) = 2^x$

Check this by using the GRAPH key on your graphing calculator.

What is special about this graph?

- First, we can see that the curve bends towards the x-axis but never touches it. In fact, even if the values of x were reduced progressively, $f(x)$ would never have the value 0.

$f(-6) =$
$f(-10) =$

We therefore say that the line $y = 0$, i.e. the x-axis, is an ***asymptote*** of the curve of the function.

A line is an asymptote of a curve when the distance between the curve and the line diminishes constantly and approaches 0 without ever reaching it.

- The curve of the function passes through the point (0, 1). The number 1 is known as the ***y-intercept***, i.e.the value of $f(x)$ when $x = 0$. We can even conclude immediately that every function of the form $f(x) = c^x$ will have 1 as its y-intercept, since $c^0 = 1$.

- The points of the function are located in quadrants I and II only, because c^x is obviously always a positive number.

- The ***domain*** of the function is $\mathbb{R}$.

- The ***range*** of the function is $]0, \infty$, since the curve does not touch the asymptote.

- This is an ***increasing*** function, since for the ordered pairs (0, 1) and (2, 4), if $x_1 < x_2$ $(0 < 2)$, then $f(x_1) \leq f(x_2)$ $(1 \leq 4)$.

Let's test these characteristics using a second example.

Example 3

Draw the curve of the function $f(x) = \left(\frac{1}{3}\right)^x$.

1. Find some of the function's ordered pairs.

$f(-3) = \left(\frac{1}{3}\right)^{-3} = 3^3 = 27$

$f(-2) = \left(\frac{1}{3}\right)^{-2} = 3^2 = 9$

$f(-1) = \left(\frac{1}{3}\right)^{-1} = 3^1 = 3$

$f(0) = \left(\frac{1}{3}\right)^{0} = 1$

$f(1) = \left(\frac{1}{3}\right)^{1} = \frac{1}{3} = 0.333$

$f(2) = \left(\frac{1}{3}\right)^{2} = \frac{1}{9} = 0.111$

$f(3) = \left(\frac{1}{3}\right)^{3} = \frac{1}{27} = 0.037$

2. Complete the table of values.

x	-3	-2	-1	0	1	2	3
$f(x)$	27	9	3	1	0.333	0.111	0.037

3. Plot these points on a grid and draw the curve.

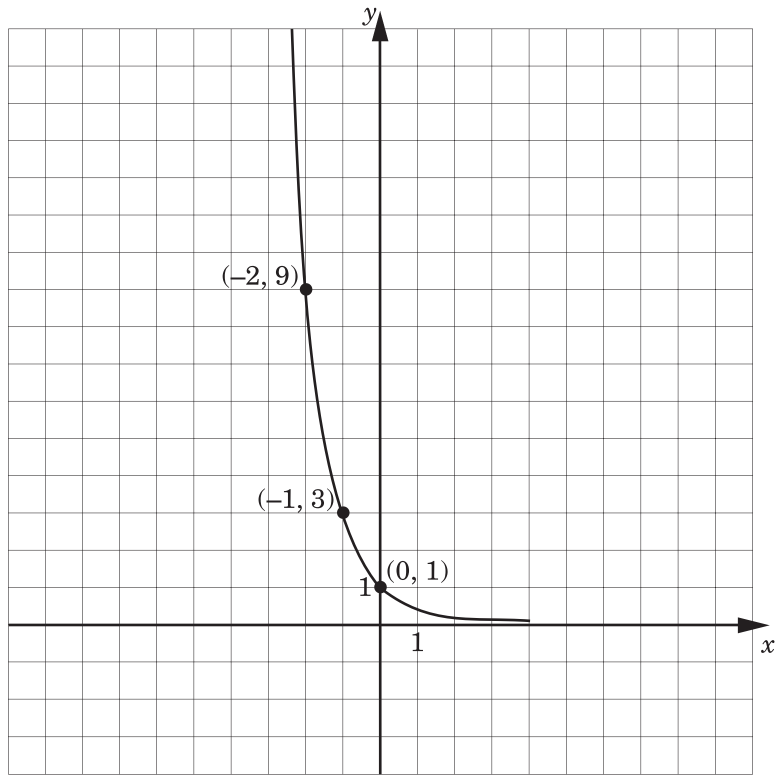


Fig. 1.2 Graph of $f(x) = \left(\frac{1}{3}\right)^x$

Can the characteristics of the preceding function apply to this latter function? Let's see!

- The asymptote is the x-axis, or the line $y = 0$.
- The curve goes through point (0, 1).
- The graph of the function is once again located in quadrants I and II.
- The domain of the function is $\mathbb{R}$.
- The range of the function is]0, ∞, since the curve does not touch the asymptote.

- This is a ***decreasing*** function, since for the ordered pairs (–2, 9) and (–1, 3), if $x_1 < x_2$ $(-2 < -1)$, then $f(x_1) \geq f(x_2)$ $(9 \geq 3)$.

As you may have noticed, the domain of an exponential function is $\mathbb{R}$ and its range is always defined according to the asymptote of its graph.

> Let an exponential function be defined by the equation $f(x) = c^x$.
>
> Dom $f = \mathbb{R}$ or $-\infty, \infty$.
>
> Ran $f = \{y \in \mathbb{R}\ y > 0\}$ or $]0, \infty$.

Let's look a little more closely at what determines whether an exponential function is increasing or decreasing.

In general, we can determine whether a function is increasing or decreasing on the basis of two of its ordered pairs.

> Let the ordered pairs be (x_1, y_1) and (x_2, y_2).
>
> - A function is an increasing function if, in every case, $x_1 < x_2 \Rightarrow y_1 \leq y_2$.
> - A function is a decreasing function if, in every case, $x_1 < x_2 \Rightarrow y_1 \geq y_2$.

Using the function graphs drawn earlier, and the other graphs below, we can establish the conditions in which exponential functions increase or decrease.

Look carefully at the following graphs.

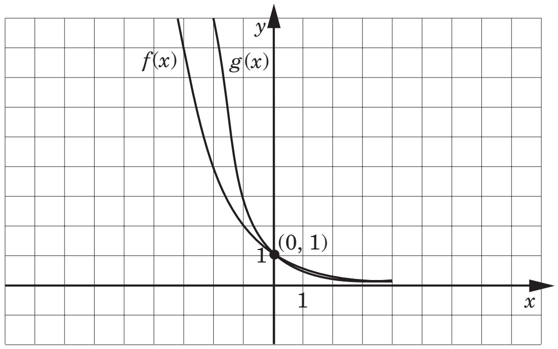


Fig. 1.3 Graph of functions $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$

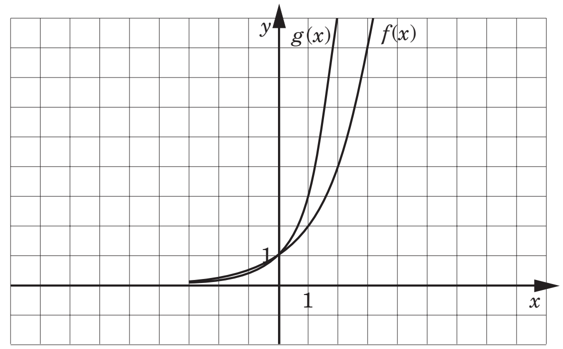


Fig. 1.4 Graph of functions $f(x) = 2^x$ and $g(x) = 3^x$

? Are the functions in Figure 1.3 increasing or decreasing functions?

....

? What can you say about the base of these functions? ....

....

Answers

These functions are decreasing, and their base is between 0 and 1.

? Are the functions in Figure 1.4 increasing or decreasing functions?

....

? What can you say about the base of these functions? ....

....

Answers

These functions are increasing, and their base is greater than 1.

Let an exponential function be defined by the equation $f(x) = c^x$.

- The function is a decreasing function if $0 < c < 1$.
- The function is an increasing function if $c > 1$.

Example 4

Let an exponential function f go through points (0, 1) and (2, 36). Let us see whether it is an increasing or a decreasing function.

Let $x_1 = 0$, $x_2 = 2$, $f(x_1) = 1$ and $f(x_2) = 36$. Since $x_1 < x_2$ $(0 < 2)$ and $f(x_1) \le f(x_2)$ $(1 \le 36)$, the function is an increasing function.

? State whether the function that goes through (0, 1) and $\left(1, \frac{1}{4}\right)$ is an increasing or a decreasing function.

....

....

....

Answer

Let $x_1 = 0, x_2 = 1, f(x_1) = 1$ and $f(x_2) = \frac{1}{4}$. Since $x_1 < x_2$ $(0 < 1)$ and $f(x_1) \ge f(x_2)$ $\left(1 \ge \frac{1}{4}\right)$, the function is a decreasing function.

Let an exponential function be defined by the equation $f(x) = c^x$. Then:

- The asymptote of the curve is the x-axis, or the line $y = 0$.
- The curve goes through point (0, 1).
- The graph is located in quadrants I and II.
- Dom $f = \mathbb{R}$.
- Ran $f =]0, \infty$.

You do not necessarily have to memorize these characteristics to draw the curve of an exponential function. However, if you do so, it will help you understand more complex exponential functions.

To draw an exponential function defined by the equation $f(x) = c^x$:

1. Find some of the ordered pairs of the function, by giving values to x and calculating $f(x)$.
2. Draw up a table of values for the ordered pairs $(x, f(x))$.
3. Plot the ordered pairs on a grid and draw the curve that joins them.

And now it's your turn!

Exercise 1.1

1. Let an exponential function be defined by $f(x) = 4^x$. Calculate the value of the following expressions. Round off your answers to the nearest thousandth, if necessary.

   a) $f(-2) =$ .... b) $f(0) =$ ....

   c) $f(1) =$ .... d) $f\left(\frac{1}{2}\right) =$ ....

2. State whether the following ordered pairs are part of the graph of the function defined by $f(x) = \exp_5 x$.

   a) (1, 5) ....

   b) (–2, –0.25) ....

   c) (0.5, 0.04) ....

   d) (0, 1) ....

   e) (–1, 0.2) ....

3. State whether the following equations are exponential functions. If not, why not?

   a) $f(x) = 10^x$ ....

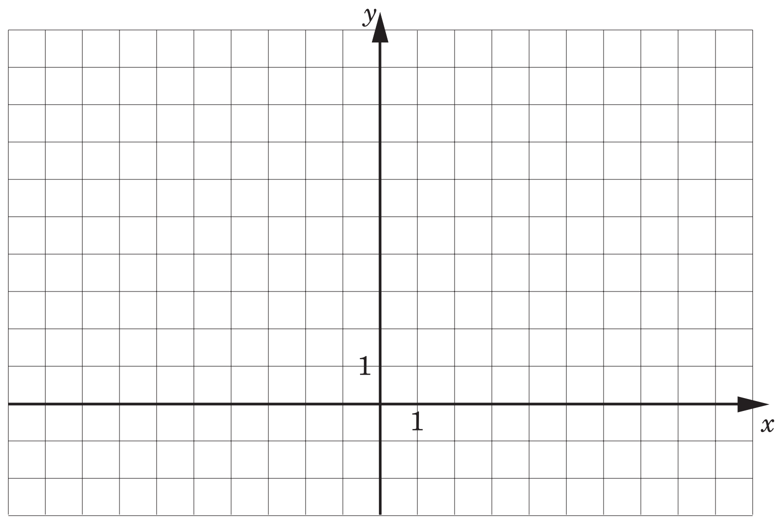
   b) $f(x) = (-10)^x$ ....

   c) $f(x) = \exp_{10} x$ ....

4. For each of the following exponential functions, first complete the table of values and then draw their graph. In addition, indicate three points on each curve, including the y-intercept.

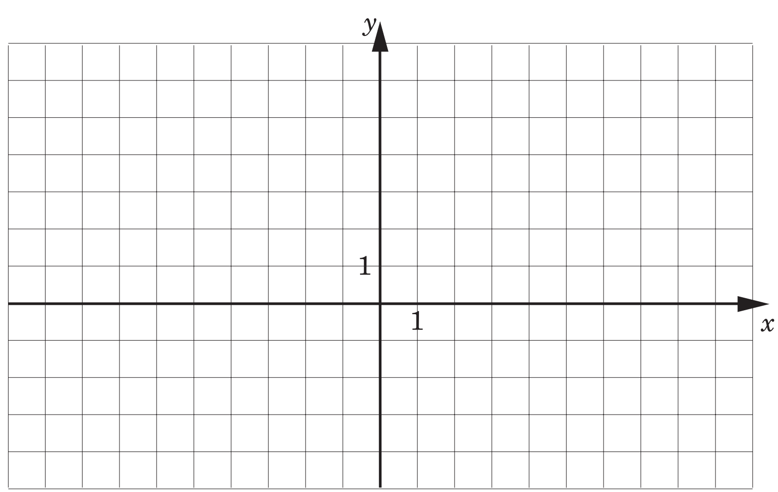
a) $f(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
$f(x)$							



b) $f(x) = \exp_3 x$

x	-3	-2	-1	0	1	2	3
$f(x)$							



5. Give the domain and the range of the following exponential functions and state whether they are increasing or decreasing functions.

a) $f(x) = 9^x$

- Dom f = ...
- Ran f = ...
- ....

b) $g(x) = \left(\frac{1}{5}\right)^x$

- Dom g = ...
- Ran g = ...
- ....

c) $h(x) = \left(\frac{4}{3}\right)^x$

- Dom h = ...
- Ran h = ...
- ....

d) $k(x) = \left(\frac{3}{4}\right)^x$

- Dom k = ...
- Ran k = ...
- ....

6. Let two functions be defined by $h(x) = \left(\frac{4}{9}\right)^x$ and by $g(x) = \left(\frac{9}{4}\right)^x$.

a) What is the domain of these functions? ....

b) What is their range? ....

c) What is the equation of their asymptote? ....

d) Which is a decreasing function? ....

e) Which is an increasing function? ....

7. Are the exponential functions that go through the points shown below increasing or decreasing functions? Give reasons for your answer.

a) $(0, 1)$ and $\left(1, \frac{3}{4}\right)$ ....

....

....

b) (0, 1) and $\left(1, \frac{6}{5}\right)$ ....

....

....

Now that you are more familiar with the graphs of exponential functions and their characteristics, we will continue our study of this type of function by adding some ***parameters***.

Reminder

A parameter is an element other than the variable x that varies in an equation. For example, in the linear function $f(x) = mx + b$, m and b are parameters.

The exponential functions we will consider next will take the form $f(x) = ac^{b(x-h)} + k$.

The following functions are exponential functions.

- $g(x) = 3^{x+2}$, where the base is 3, $h = -2$ and $k = 0$.
- $h(x) = 5^{x-1} + 2$, where the base is 5, $h = 1$ and $k = 2$.
- $k(x) = -2 \bullet 3^x - 5$ where the base is 3, $a = -2$, $h = 0$ and $k = -5$.
- $f(x) = 4^{3(x-2)} - 1$, where the base is 4, $b = 3$, $h = 2$ and $k = -1$.

We will study these functions by comparing them with the exponential form we already know, i.e. $f(x) = c^x$.

As is the case with all the other functions, the basic form of the function can be altered. For instance, we can apply a ***translation*** or a scaling factor to it. The standard form of an exponential function is a good example of a basic form that has undergone a transformation.

The standard form of an exponential function $f(x) = ac^{b(x-h)} + k$ keeps the same base c but the exponent becomes $b(x - h)$.

In this form, we added the parameters a, b, h and k, which all have a specific effect on the curve. It is important that you be able to recognize and understand the influence of these parameters.

Let us begin by seeing what changes occur in our curve following the addition of parameter a.

Role of Parameter a in the Exponential Function

Example 5

Sketch the function defined by $g(x) = 2 \bullet 2^x$ and compare its graph with that of $f(x) = 2^x$.

1. Find some ordered pairs $(x, g(x))$.
   - $g(-2) = 2 \times 2^{-2} = 2 \times \frac{1}{4} = \frac{1}{2} = 0.5$
   - $g(-1) = 2 \times 2^{-1} = 2 \times \frac{1}{2} = 1$
   - $g(0) = 2 \times 2^0 = 2 \times 1 = 2$

   - $g(1) =$ ....
   - $g(2) =$ ....
   - $g(3) =$ ....

2. Complete the table of values.

x	-2	-1	0	1	2	3
$g(x)$	0.5	1	2	4	8	16

3. Plot the points on a grid, draw a smooth line through them and compare $f(x)$ and $g(x)$.

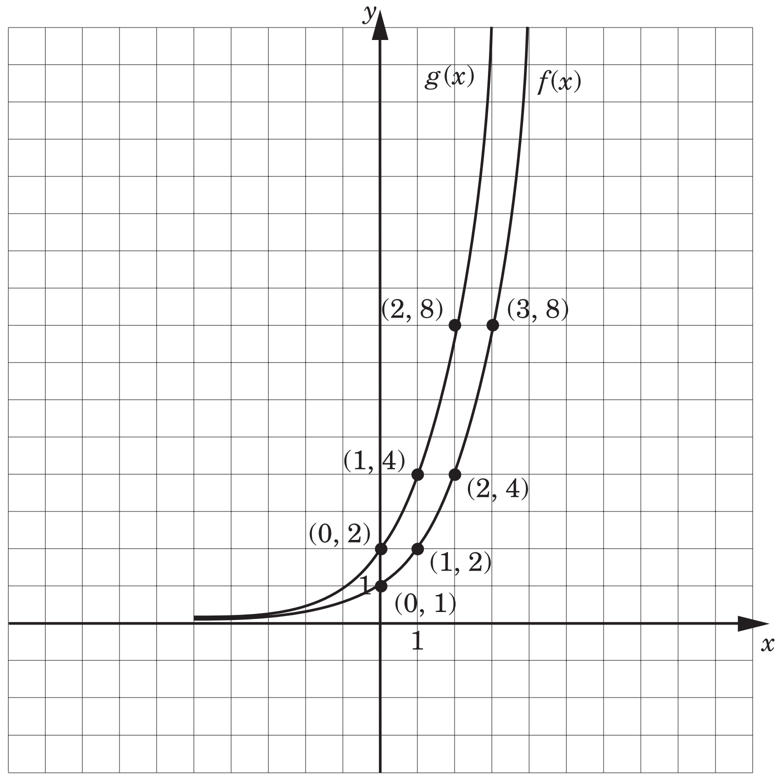


Fig. 1.5 Comparison of the graphs of $f(x) = 2^x$ and $g(x) = 2 \bullet 2^x$

The only notable change is this: the curve of $g(x)$ is the same as that of $f(x)$, but has shifted upwards. We therefore say that the curve has undergone a vertical scale change.

? What do you notice by comparing the coordinates of the two curves?

....

....

....

....

....

(0, 1) is now $(0, 2 \times 1) = (0, 2)$.

(1, 2) is now $(1, 2 \times 2) = (1, 4)$.

(2, 4) is now $(2, 2 \times 4) = (2, 8)$.

The ordinate of the ordered pairs of $f(x)$ is multiplied by 2, or the value of a. The ordered pairs (x, y) of $f(x)$ become (x, ay) in $g(x)$.

Now let's look at what happens when we assign a negative value to a.

Example 6

Sketch the function defined by $h(x) = -2 \bullet 2^x$ and compare its graph with that of $g(x) = 2 \bullet 2^x$.

1. Find some ordered pairs $(x, h(x))$.

- $h(-2) = -2 \times 2^{-2} = -2 \times \frac{1}{4} = -\frac{1}{2} = -0.5$
- $h(-1) = -2 \times 2^{-1} = -2 \times \frac{1}{2} = -1$
- $h(0) = -2 \times 2^0 = -2 \times 1 = -2$

? • $h(1) =$ ....

? • $h(2) =$ ....

? • $h(3) =$ ....

2. Complete the table of values.

x	-2	-1	0	1	2	3
$h(x)$	-0.5	-1	-2	-4	-8	-16

3. Plot these points on a grid, draw a smooth line through them and compare $g(x)$ and $h(x)$.

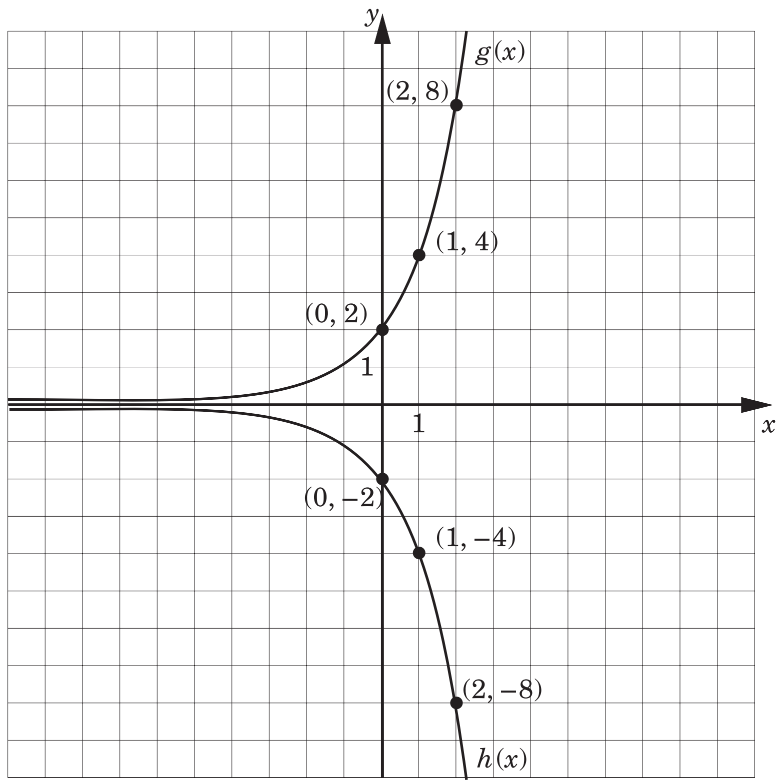


Fig. 1.6 Comparison of the graphs of $g(x) = 2 \bullet 2^x$ and $h(x) = -2 \bullet 2^x$

The only notable change is this: the curve of $h(x)$ is the same as that of $g(x)$ but it is located below the x-axis. We therefore say that the curve has undergone a reflection over the asymptote and an inversion of its growth (i.e., increasing curves become decreasing and decreasing curves become increasing).

Role of Parameter *b* in the Exponential Function

Example 7

Sketch the function defined by $i(x) = 2^{2x}$ and compare its graph with that of $f(x) = 2^x$.

1. Complete the table of values using a graphing calculator.

x	-2	-1	0	1	2	3
$i(x)$	0.063	0.25	1	4	16	64

2. Plot these points on a grid, draw a smooth line through them and compare $f(x)$ and $h(x)$.

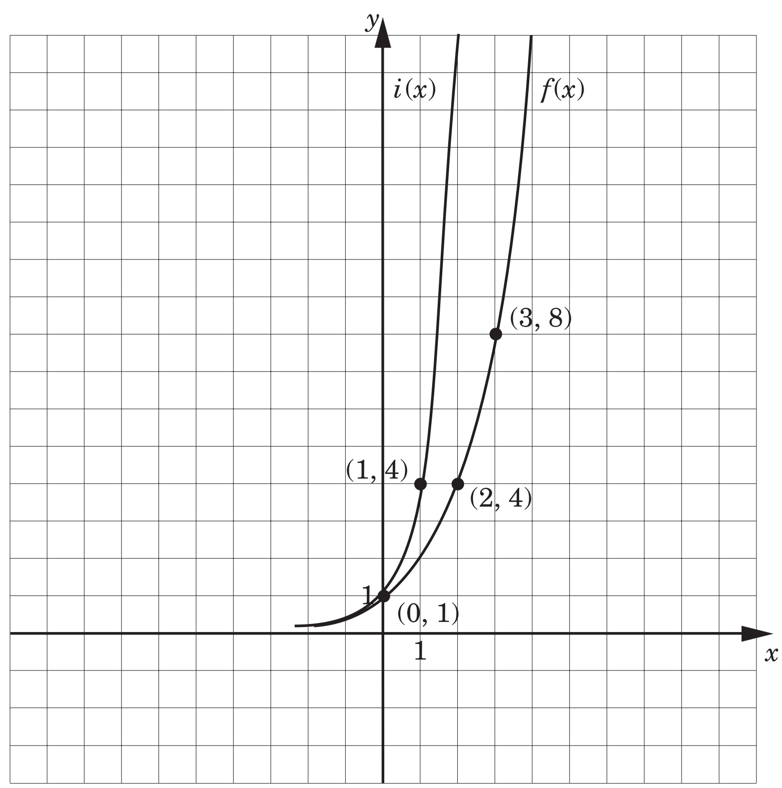


Fig. 1.7 Comparison of the graphs of $f(x) = 2^x$ and $i(x) = 2^{2x}$

Note that the curve has undergone a horizontal scale change, that is, the abscissas of the ordered pairs have been divided by 2. In fact,
(0, 1) becomes (0 ÷ 2; 1) = (0, 1),
(2, 4) becomes (2 ÷ 2; 4) = (1, 4),
(4, 16) becomes (4 ÷ 2, 16) = (2, 16).

The ordered pairs (x, y) of $f(x)$ become $\left(\frac{x}{b}, y\right)$ in $i(x)$.

Now, let's see what happens when we assign a negative value to b.

Example 8

Sketch the function defined by $j(x) = 2^{-2x}$ and compare its graph with that of $i(x) = 2^{2x}$.

1. Complete the table of values using a graphing calculator.

x	−2	−1	0	1	2	3
$j(x)$	16	4	1	0.25	0.063	0.016

2. Plot these points on a grid, draw a smooth line through them and compare $i(x)$ and $j(x)$.

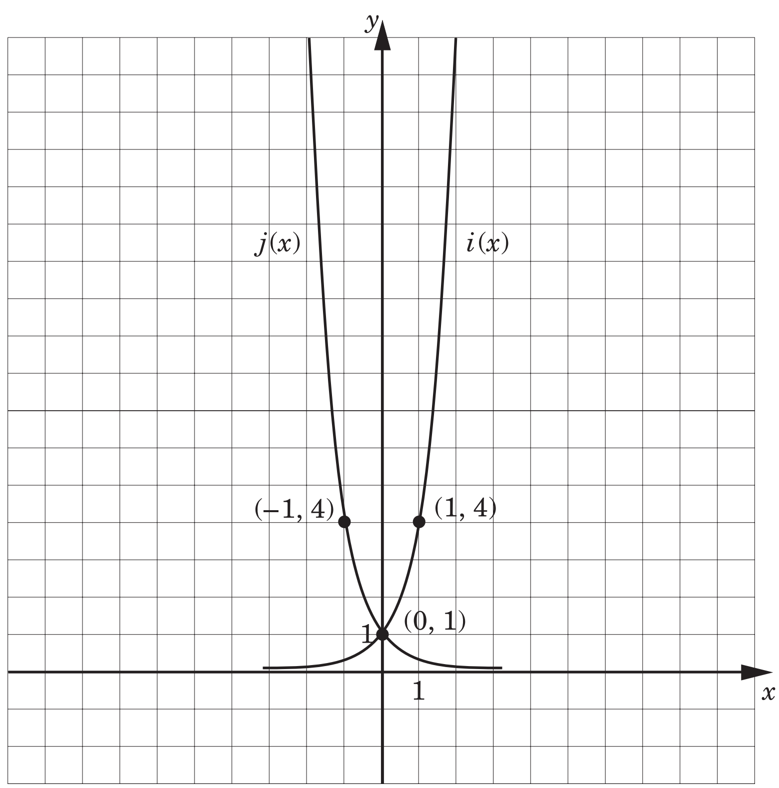


Fig. 1.8 Comparison of the graphs of $i(x) = 2^{2x}$ and $j(x) = 2^{-2x}$

The only notable change is this: the curve of $j(x)$ is the same as that of $i(x)$, but it has undergone a reflection over the vertical axis (y-axis) and an inversion of its growth (i.e., increasing curves become decreasing and decreasing curves become increasing).

Role of Parameter *h* in the Exponential Function

Example 9

Sketch the function defined by $k(x) = 2^{x-3}$ and compare its graph with that of $f(x) = 2^x$.

1. Complete the table of values using a graphing calculator.

x	0	1	2	3	4	5	6
$k(x)$	0.125	0.25	0.5	1	2	4	8

2. Plot these points on a grid, draw a smooth line through them and compare $f(x)$ and $i(x)$.

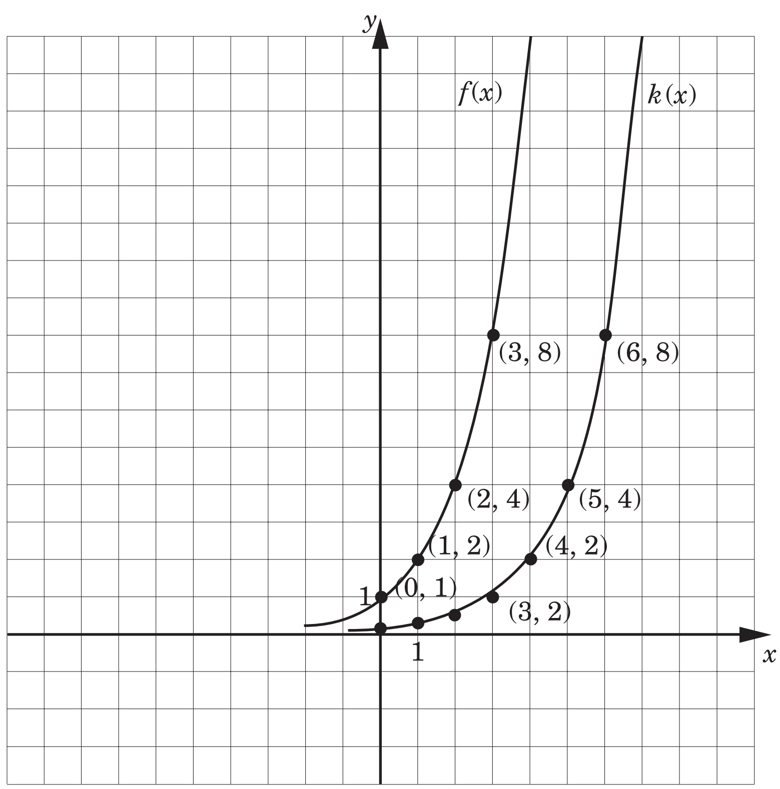


Fig. 1.9 Comparison of the graphs of $f(x) = 2^x$ and $k(x) = 2^{x-3}$

Note that the curve has undergone a horizontal translation of 3 units to the right. In fact,
(0, 1) becomes (0 + 3, 1) = (3, 1),
(1, 2) becomes (1 + 3, 2) = (4, 2),
(2, 4) becomes (2 + 3, 4) = (5, 4).

The ordered pairs (x, y) of $f(x)$ become $(x + h, y)$ in $k(x)$.

ROLE OF PARAMETER *k* IN THE EXPONENTIAL FUNCTION

Example 10

Sketch the function defined by $l(x) = 2^x - 3$ and compare its graph with that of $f(x) = 2^x$.

1. Complete the table of values using a graphing calculator.

x	-2	-1	0	1	2	3	4
$l(x)$	-2.75	-2.5	-2	-1	1	5	13

2. Plot these points on a grid, draw a smooth line through them and compare $f(x)$ and $l(x)$.

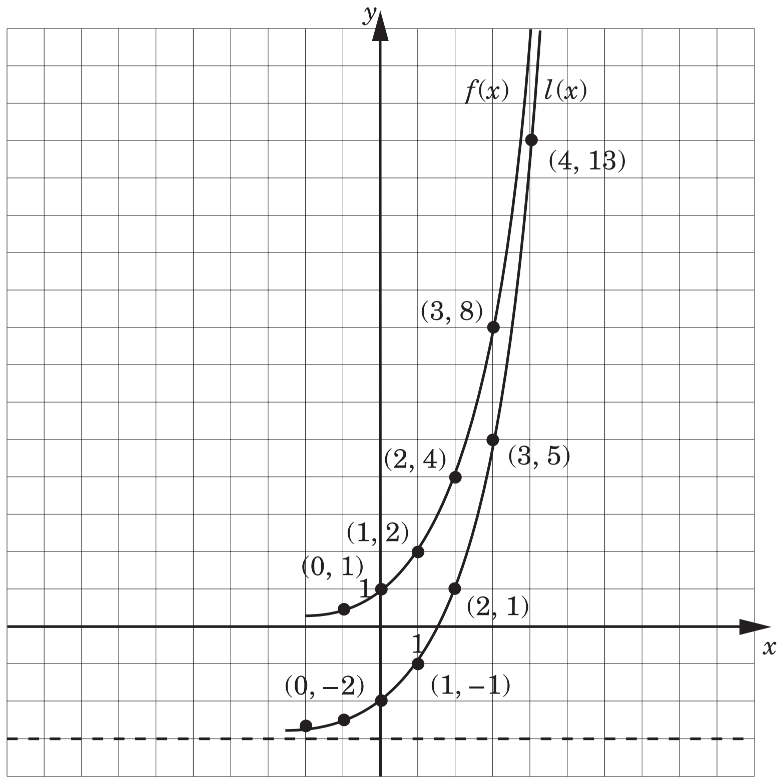


Fig. 1.10 Comparison of the graphs of $f(x) = 2^x$ and $l(x) = 2^x - 3$

Note that the curve has undergone a vertical translation of 3 units downwards. In fact,

(0, 1) becomes (0, 1 – 3) = (0, –2),

(1, 2) becomes (1, 2 – 3) = (1, –1),

(2, 4) becomes (2, 4 – 3) = (2, 1).

The ordered pairs (x, y) of $f(x)$ become $(x, y + k)$ in $l(x)$.

Now let's summarize the role of each parameter.

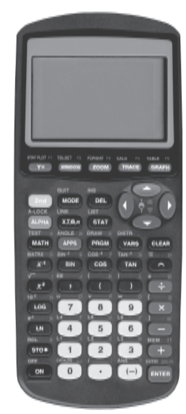
If $f(x) = c^x$ is the basic form of the exponential function and $g(x) = ac^{b(x-h)} + k$ is the standard form, then the ordered pairs (x, y) of $f(x)$ become $\left(\frac{x}{b} + h, ay + k\right)$ in $g(x)$ where:

- a imposes a vertical scale change on the curve; if a is negative, the curve undergoes a reflection over the asymptote and an inversion of its growth (i.e., increasing curves become decreasing and decreasing curves become increasing).
- b imposes a horizontal scale change on the curve; if b is negative, the curve undergoes a reflection over the y-axis and an inversion of its growth (i.e., increasing curves become decreasing and decreasing curves become increasing).
- h imposes a horizontal translation, or shift, on the curve.
- k imposes a vertical translation, or shift, on the curve.

After having systematically studied the role of the different parameters of the exponential function, let's look at some of its characteristics.

Example 11

Sketch the graph of the exponential function $f(x) = 2 \bullet 2^{2(x-2)} - 5$ using your graphing calculator.



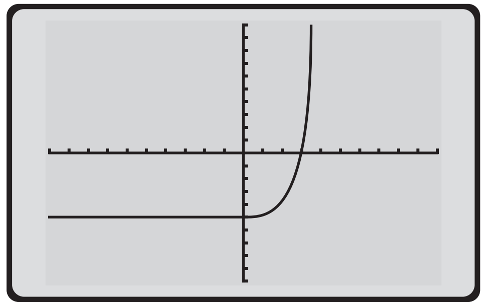
$\boxed{\text{Y=}}$ $\boxed{\text{CLEAR}}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{2}$ $\boxed{\wedge}$ $\boxed{(}$ $\boxed{2}$ $\boxed{(}$ $\boxed{\text{X, T, Ø, } n}$ $\boxed{-}$ $\boxed{2}$ $\boxed{)}$ $\boxed{)}$ $\boxed{-}$ $\boxed{5}$

$\boxed{\text{2nd}}$ $\boxed{\text{TABLE}}$ (GRAPH)

X	Y1	
-2	-4.992	
-1	-4.969	
0	-4.875	
1	-4.5	
2	-3	
3	3	
4	27	

X=-2

GRAPH



Determine the following characteristics of the curve.

- Equation of the asymptote: ....
- Dom f = ...
- Ran f = ...
- y-intercept: ....
- Increasing or decreasing, with an explanation: ....

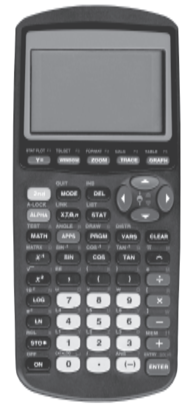
  ....

Solution

This function has an asymptote at $y = -5$ ($k = -5$), its domain is $\mathbb{R}$, its range is $]-5, \infty$ and its y-intercept is -4.875. It is an increasing function. Given the points $(2, -3)$ and $(3, 3)$: $x_1 < x_2$ $(2 < 3) \Rightarrow f(x_1) \leq f(x_2)$ $(-3 \leq 3)$.

Example 12

Sketch the graph of the exponential function $g(x) = -2 \bullet \left(\frac{1}{2}\right)^{\frac{1}{2}(x+1)} + 3$ using your graphing calculator.



Y= CLEAR (–) 2 × 0.5 ∧ (0.5 (X, T, Ø, n + 1)) + 3

2nd TABLE (GRAPH)

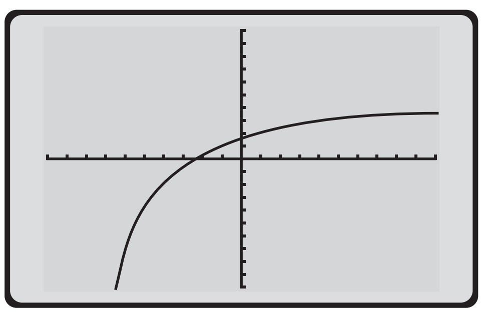
X	Y_1
-7	-13
-6	-8.314
-5	-5
-4	-2.657
-3	-1
-2	.17157
-1	1

X=-7

X	Y_1
0	1.5858
1	2
2	2.2929
3	2.5
4	2.6464
5	2.75
6	2.8232

X=6

GRAPH



? Determine the following characteristics of the curve.

- Equation of the asymptote: ....
- Dom g = ...
- Ran g = ...
- y-intercept: ....

- Increasing or decreasing, with an explanation: ....

  ....

Solution

This function has an asymptote at $y = 3$ ($k = 3$), its domain is $\mathbb{R}$, its range is $-\infty, 3[$ and its y-intercept is 1.5858. It is an increasing function. Given the points $(-1, 1)$ and $(1, 2)$: $x_1 < x_2$ $(-1 < 1) \Rightarrow f(x_1) \leq f(x_2)$ $(1 \leq 2)$.

Let's compare the different characteristics of the curves obtained in the two examples. The equation of each asymptote is determined by the parameter k. When $a > 0$, the curve is located above the asymptote and when it $a < 0$, it is located below the asymptote. The range of f is $]k, \infty$ and that of g is $-\infty, k[$. Both are increasing functions in $\mathbb{R}$.

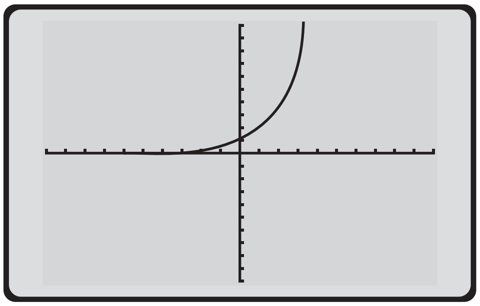
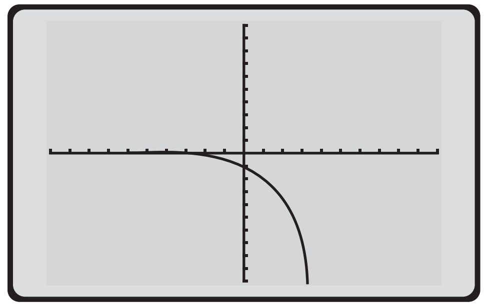
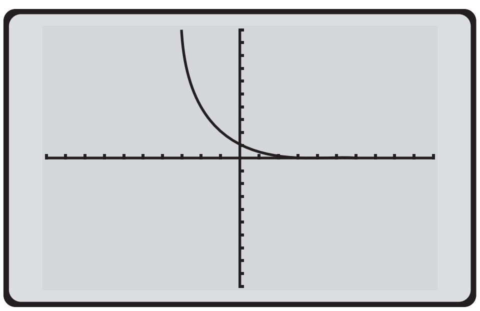
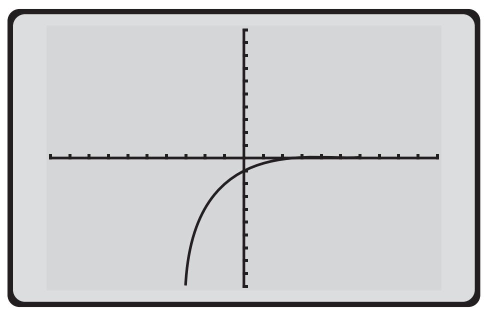
Observations

We cannot always precisely determine the sign of the function when $k \neq 0$, since we have not yet looked at the concept of logarithm. However, when $k \neq 0$, the function is positive if $a > 0$ and negative if $a < 0$.

A positive exponential function is not necessarily an increasing function and a negative exponential function is not necessarily a decreasing function. The three parameters a, b and c play a role in determining whether a function is an increasing or a decreasing function.

To determine whether an exponential function is an increasing or a decreasing function, ask yourself the following three questions: is the function positive ($a > 0$) or negative ($a < 0$), is b positive ($b > 0$) or negative ($b < 0$) and is $c > 1$ or $0 < c < 1$?

To better understand the role the parameters play, study the graphs below.

	$a > 0, b > 0$	$a < 0, b > 0$
$c > 1$	$f(x) = 2^x$  Increasing and positive	$f(x) = -(2)^x$  Decreasing and negative
$0 < c < 1$	$f(x) = \left(\frac{1}{2}\right)^x$  Decreasing and positive	$f(x) = -\left(\frac{1}{2}\right)^x$  Increasing and negative

Let's summarize the situation

	Positive $a > 0, b > 0$	**Negative** $a < 0, b > 0$
$c > 1$	• Equation of the asymptote: $y = k$. • Domain: $\mathbb{R}$. • Range: $]k, \infty$. • Sign: positive if $k \geq 0$. • y-intercept: $ac^{b(-h)} + k$. • Increasing, since for the ordered pairs (x_1, y_1) and (x_2, y_2), $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.	• Equation of the asymptote: $y = k$. • Domain: $\mathbb{R}$. • Range: $-\infty, k[$. • Sign: negative if $k \leq 0$. • y-intercept: $ac^{b(-h)} + k$. • Decreasing, since for the ordered pairs (x_1, y_1) and (x_2, y_2), $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2, y_2)$.
$0 < c < 1$	• Equation of the asymptote: $y = k$. • Domain: $\mathbb{R}$. • Range: $]k, \infty$. • Sign: positive if $k \geq 0$. • y-intercept: $ac^{b(-h)} + k$. • Decreasing, since for the ordered pairs (x_1, y_1) and (x_2, y_2), $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.	• Equation of the asymptote: $y = k$. • Domain: $\mathbb{R}$. • Range: $-\infty, k[$. • Sign: negative if $k \leq 0$. • y-intercept: $ac^{b(-h)} + k$. • Increasing, since for the ordered pairs (x_1, y_1) and (x_2, y_2), $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Observations

In the above table, parameter b is positive.

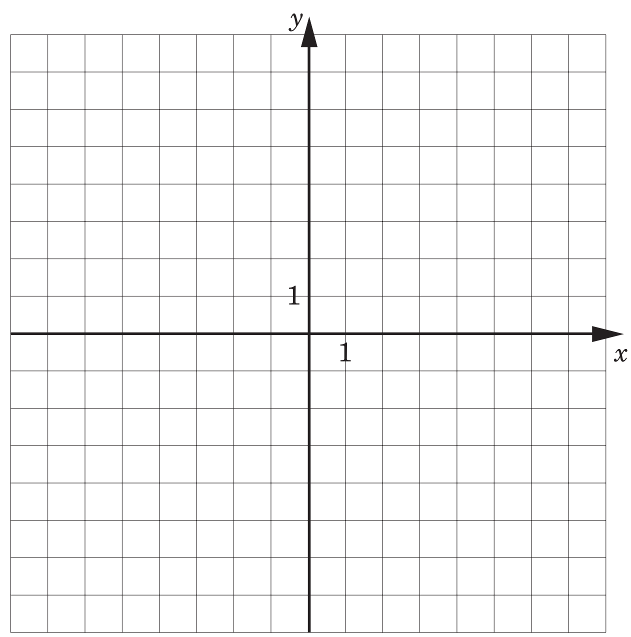
If we change the sign of parameter b ($b < 0$), the curve undergoes a reflection over the y-axis (when $h = 0$) and an inversion of its growth (i.e., increasing curves become decreasing and decreasing curves become increasing).

Exercise 1.2

1. Draw the graphs of the following exponential functions. Remember to draw the asymptote as a dotted line, and to identify the y-intercept and two other points of the curve.

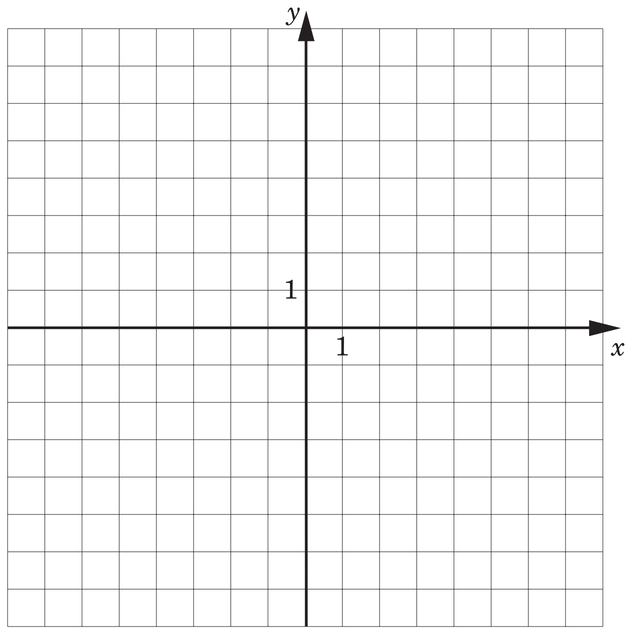
a) $g(x) = 3^{x-1}$

x							
$g(x)$							



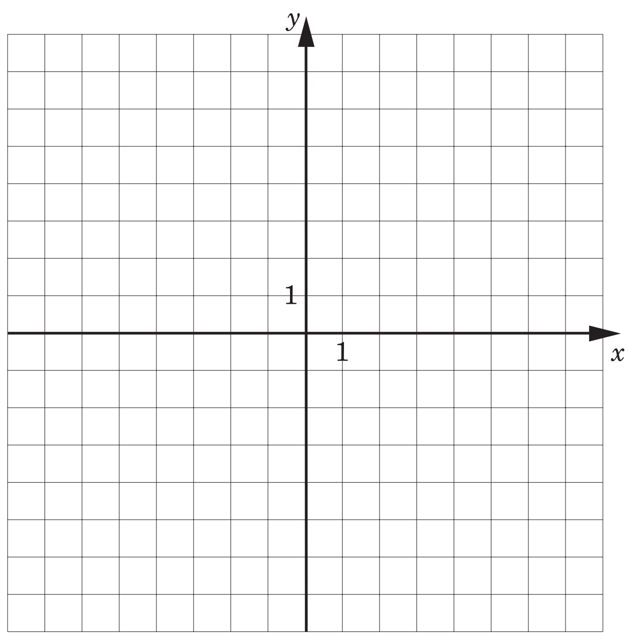
b) $h(x) = \left(\frac{2}{3}\right)^x + 2$

x							
$h(x)$							



c) $k(x) = 2^{x+1} + 4$

x							
$k(x)$							



2. Give the following characteristics of the functions shown below, but do not map them.
   1. Whether it is an increasing or a decreasing function.
   2. The domain.
   3. The range.
   4. The equation of its asymptote.

a) $g(x) = \left(\frac{6}{5}\right)^x + \frac{1}{2}$

1. ....
2. ....
3. ....
4. ....

b) $h(x) = 7^{x-5} - 6$

1. ....
2. ....
3. ....
4. ....

c) $k(x) = 10^{x+4}$

1. ....
2. ....
3. ....
4. ....

d) $p(x) = \left(\frac{2}{3}\right)^x - 3$

1. ....
2. ....
3. ....
4. ....

3. A function f is defined by the equation $f(x) = 5^x$.

   a) If g is a function obtained by shifting function f 6 units to the right, what is the equation of g? ....

   b) Calculate $g(2)$. ....

   c) If h is a function obtained by shifting function f 8 units downwards, what is the equation of h? ....

   d) What is the equation of the asymptote of function h? ....

   e) Give the domain and range of h. ....

f) If i is a function obtained by imposing a vertical scaling factor of 3 on function f, what is the equation of i?

....

g) If j is a function obtained by imposing a horizontal scaling factor of 2 on function f, what is the equation of j?

....

4. The coordinates of two points of an exponential function are (0, 1) and (3, 27). Find the new coordinates following a change in one of the parameters.

a) If $a = 2$, (0, 1) becomes .... and (3, 27) becomes ....

b) If $k = 3$, (0, 1) becomes .... and (3, 27) becomes ....

c) If $b = 5$, (0, 1) becomes .... and (3, 27) becomes ....

d) If $h = 6$, (0, 1) becomes .... and (3, 27) becomes ....

e) If $a = -4$, (0, 1) becomes .... and (3, 27) becomes ....

Did you know that...

... we won't ask you to count to a billion? Don't worry, if you counted at a rate of 1 number every 2 seconds, it would take you more than 63 years, and you surely have better things to do with your life! Yet, compared to 1 million (less than 2^{20}), 1 billion is slightly less than 2^{30}!

To form a better idea of the number, suppose we were to pile up 1 billion dollars in \$10 bills, and then compare the height of the pile with the height of Place Ville-Marie in Montréal.

? How many stories would the pile reach, or by how many stories would it exceed Place Ville-Marie?

....

....

If you thought the pile would be smaller than Place Ville-Marie, you weren't even close! Do a quick calculation, based on the fact that Place Ville-Marie is 188 m high, and a \$10 bill is 0.14 mm thick.

....

....

....

....

....

....

Solution

\$1 000 000 000 ÷ \$10 = 100 000 000 \$10 bills
0.14 mm = 0.000 14 m
100 000 000 × 0.000 14 m = 14 000 m high
14 000 m ÷ 188 m ≈ 75 times the height of Place Ville-Marie

1.2 PRACTICE EXERCISES

1. Consider the functions defined by the following equations.
   - $f(x) = 4^x$
   - $g(x) = 4^{x+1}$
   - $h(x) = 4^x + 3$
   - $i(x) = 4^{2x}$
   - $j(x) = -2 \bullet 4^x + 2$
   - $k(x) = 4^{x+1} + 3$
   - $l(x) = \left(\frac{1}{4}\right)^x$

   a) Complete the following sentences.

   Function f goes through the point of abscissa 0 and ordinate .... The graph of this function is located in quadrants .... Using function f as a basis, function g is obtained by a .... of .... unit(s) to the ...., while h is obtained by a .... of .... unit(s) .... Still using f as a basis, function k is obtained by a .... of .... unit(s) to the .... and of .... unit(s) ....

   The graph of function j is located in quadrants .... The sign of functions i and l is .... and that of function j in the interval $[0, \infty$ is .... The base of function l is .... If (0, 1) and (2, 16) are ordered pairs of function f, in function i these ordered pairs become .... and .... and in function j, they become .... and ....

b) What is the domain of all these functions?

c) What is the range of each function?

Ran f = ...

Ran g = ...

Ran h = ...

Ran i = ...

Ran j = ...

Ran k = ...

Ran l = ...

d) State whether they are increasing or decreasing functions.

....

....

e) Give the equation of the asymptote of the functions.

f: ....; g: ....; h: ....; i: ....;

j: ....; k: ....; l: ....

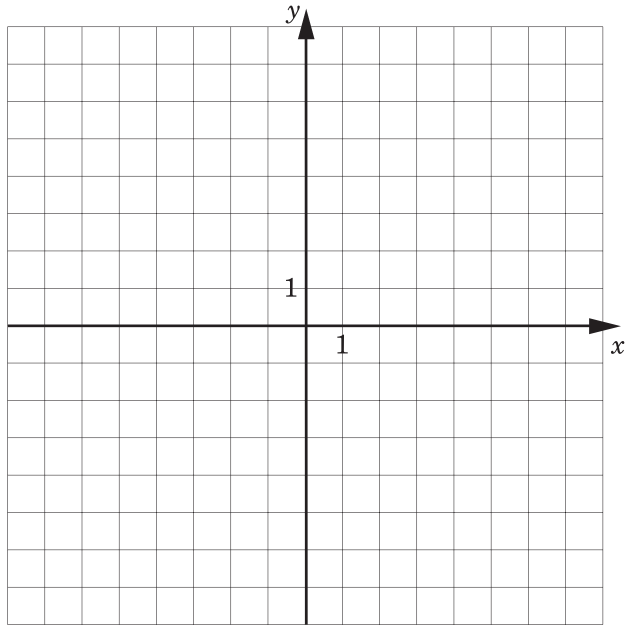
f) Find the range of 2 under g: ....

g) Complete the following tables for functions f and k.

x	−3	−2	−1	0	1	2	3
$f(x)$							

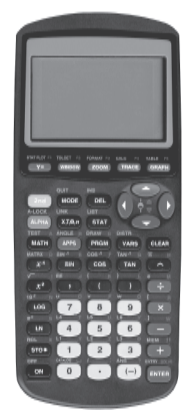
x	−3	−2	−1	0	1	2	3
$k(x)$							

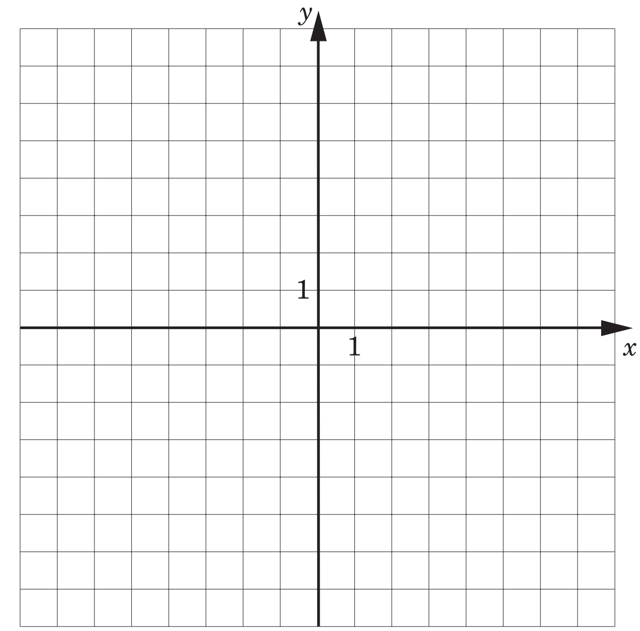
h) Draw functions f and k in the same grid, identifying three points of each curve on the graph, including the y-intercept, and their asymptotes.



2. a) Graph the exponential function $f(x) = -\left(\frac{4}{5}\right)^{2(x-4)} + 7$.

x	-2	-1	0	1	2	4	6
$f(x)$							





b) What is the domain of f? ....

c) What is the range of f? ....

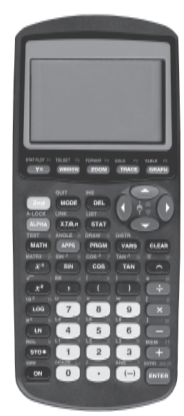
d) What is the equation of the asymptote? ....

e) What is the y-intercept? ....

f) Is f an increasing or a decreasing function? Explain. ....

....

3. Consider the equation of function f defined by $f(x) = -3^{2x+4} + 5$.



a) What is the domain of f? ....

b) What is the range of f? ....

c) What is the equation of the asymptote? ....

d) What is the y-intercept? ....

e) Is f an increasing or a decreasing function? Explain.

....

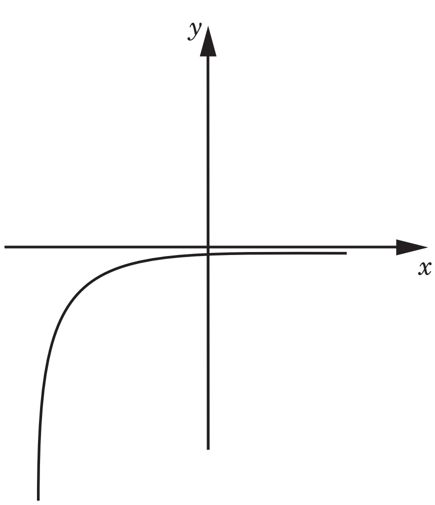
....

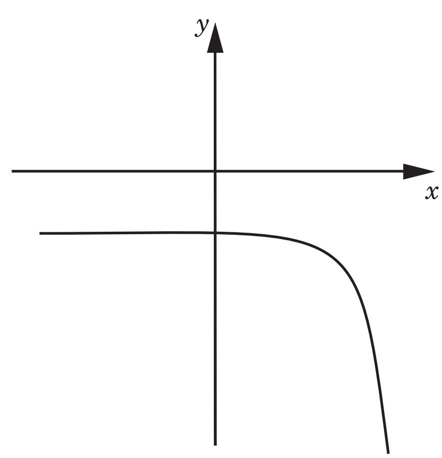
f) If f undergoes a translation of $(3, 2)$, what is the new rule of this function?

....

4. Given function g of the form $g(x) = c^x$ and function h of the form $h(x) = ac^{bx} + k$ as well as their respective graphs below, determine the parameters (a, b or k) that changed and state how.
N.B. Parameters a and b can take positive or negative values and the value of parameter k can increase or decrease.

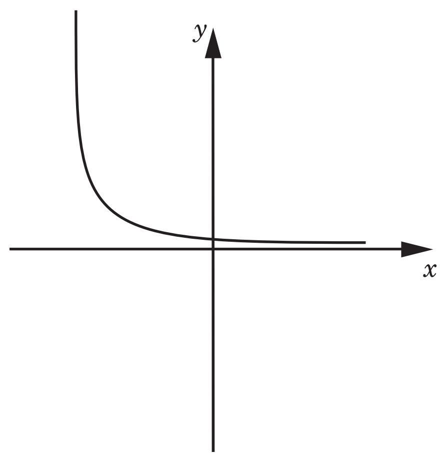
a) $g(x) = c^x$ $h(x) = ac^{bx} + k$

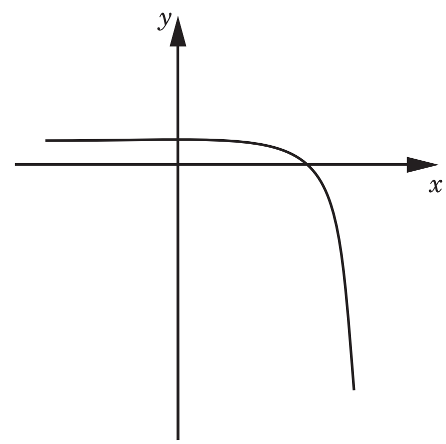




....

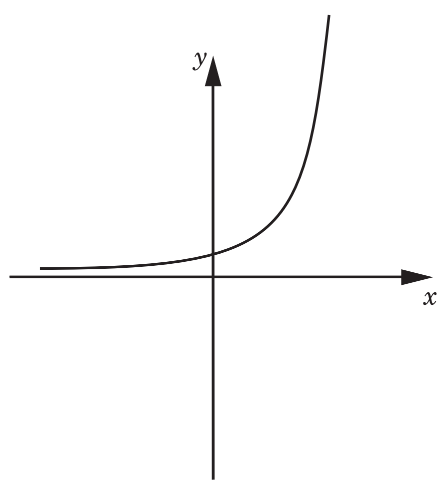
b) $g(x) = c^x$ $h(x) = ac^{bx} + k$

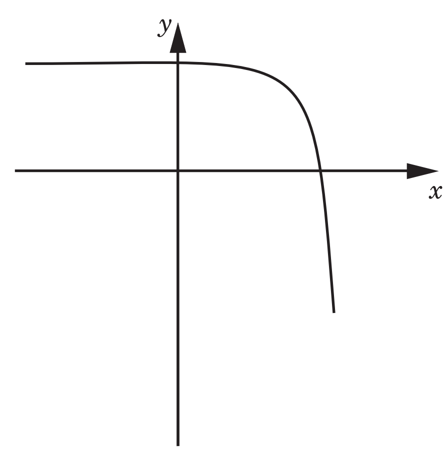




....

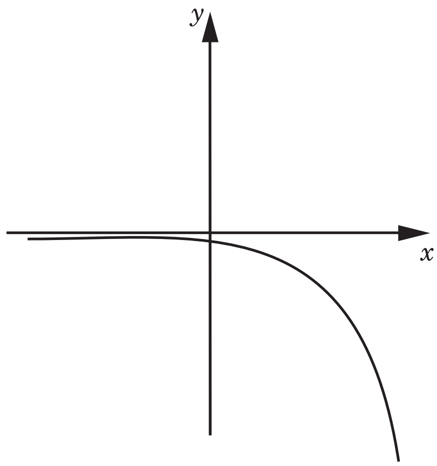
c) $g(x) = c^x$ $h(x) = ac^{bx} + k$

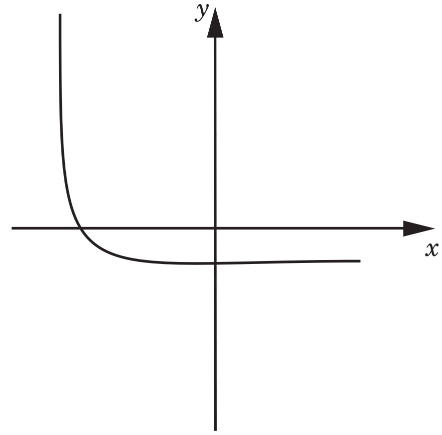




....

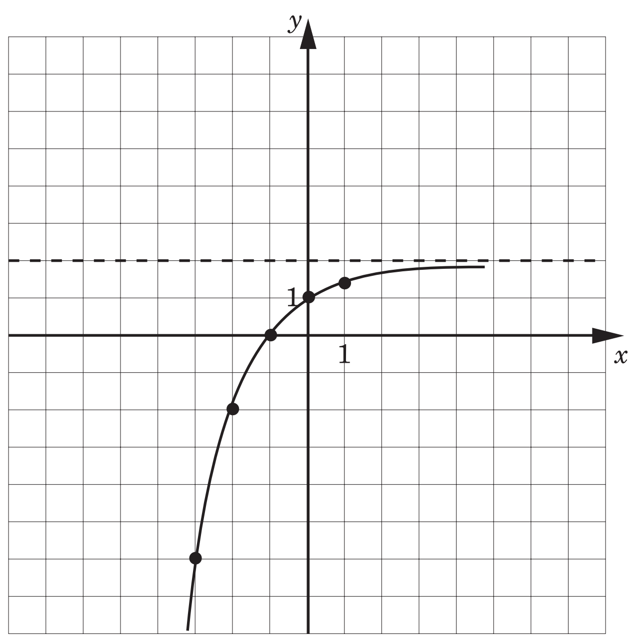
d) $g(x) = c^x$ $h(x) = ac^{bx} + k$





....

5. The graph of an exponential function g defined by the rule $g(x) = ac^x + k$ is shown below.

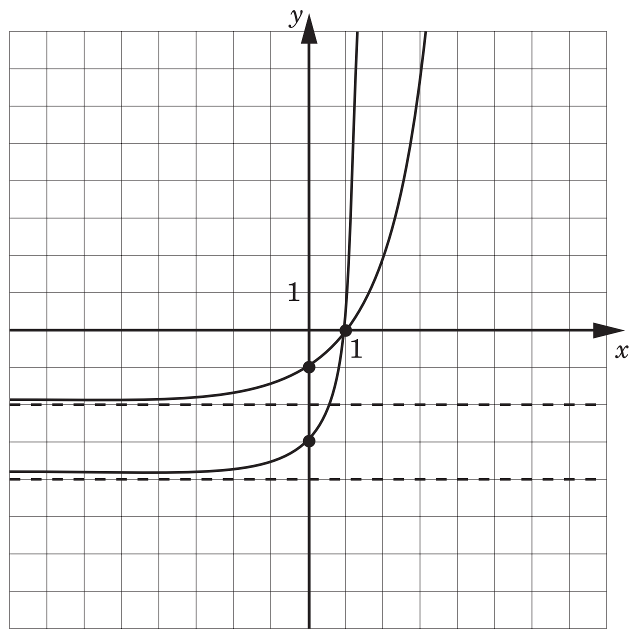


State whether $a > 0$ or whether $a < 0$, if $c > 1$ or if $0 < c < 1$ and if $k < 0$ or $k > 0$.

....

6. The graph of the following two functions is shown below.

$$h(x) = 2^{2x} - 4$$
$$k(x) = 2^{x} - 2$$



a) What is the range of each function?

....

b) Determine the y-intercept of each function.

....

c) What are the coordinates of the point of intersection of the two functions?

....

1.3 REVIEW EXERCISES

In this unit, we discovered a new real function, known as the exponential function. Its most elaborate form is $f(x) = ac^{b(x-h)} + k$. To understand the role of variable c and parameters a, b, h and k, we can compare this function with the function $f(x) = c^x$.

1. What information does variable c provide in an exponential function?

   ....

   ....

2. If $k > 0$ in an exponential function g, how does function g behave in comparison with f?

   ....

3. Let $k = 2$ in function g.

   a) What is the equation of the asymptote of g? ....

   b) What is the range of g? ....

4. If $a > 0$ and $k = 0$, in which quadrants is function g? ....

5. If $a < 0$ and $k = 0$, in which quadrants is function g? ....

6. Function h has ordered pairs (2, 3) and (3, 9) and undergoes a vertical translation of 2 units.

   a) Which parameter (a, b, h, k) changed? ....

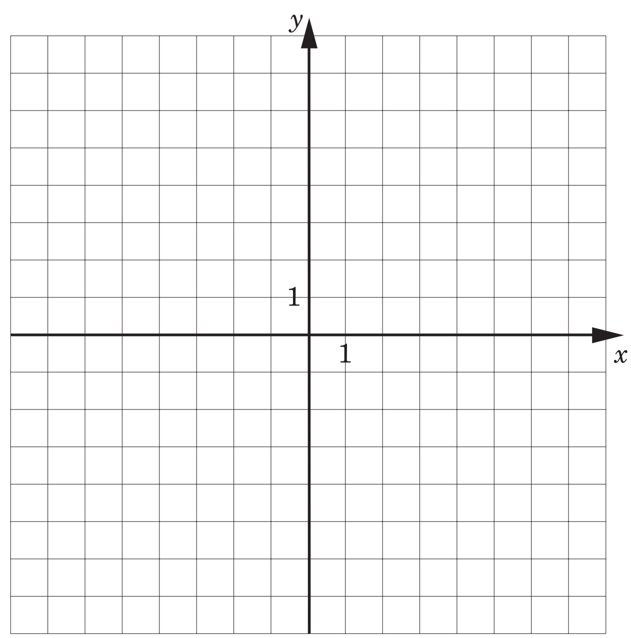
   b) What are the new coordinates of h?

   ....

   ....

7. Draw a graph of the function defined by $g(x) = \left(\frac{1}{2}\right)^{x-1} + 2$.

x							
$f(x)$							



1.4 THE MATH WHIZ PAGE

Exponential Equations for Everyone!

Exponential equations are easy and fun to solve, provided you know the laws of exponents and one property of exponents that you will have no difficulty understanding.

If $b^c = b^d$, then $c = d$.

Let us look at this property in the following examples.

- If $4^x = 4^3$, then $x = 3$.

- If $2^{x+1} = 2^{2x}$, then $x + 1 = 2x$.

$$x - 2x = -1$$
$$-x = -1$$
$$x = 1$$

- If $2^x = 16$, then $2^x = 2^4$.

$$x = 4$$

The secret: make the bases equal. Now let's complicate matters.

Solve $27^x \times 81^{x-2} = 9$.

$$(3^3)^x \times (3^4)^{x-2} = 3^2$$

$$3^{3x} \times 3^{4x-8} = 3^2$$

$$3^{3x+4x-8} = 3^2$$

$$3^{7x-8} = 3^2$$

$$7x - 8 = 2$$

$$7x = 10$$

$$x = \frac{10}{7}$$

Now it's time for you to put your new knowledge to the test!

Solve the following exponential equations.

1. $5^{x-1}(25^{2x+2}) = 125^{x}$

2. $\dfrac{1\,000^{x}}{10^{4x-3}} = \dfrac{1}{10}$

3. $16^{4x-3}(4^{x+1}) - 1 = 0$