MTH-4102-1

Geometry III

sofad

MTH-4102-1

GEOMETRY III

sofad

This course was produced by the Direction générale de la formation à distance of the ministère de l'Éducation du Québec in collaboration with the Service de l'éducation des adultes de la Commission scolaire catholique de Sherbrooke and the Departement of the Secretary of State of Canada.

Authors: Jacques Gravel
Suzie Asselin

Content revision: Jean-Paul Groleau
Daniel Gélineau
Mireille Moisan-Sanscartier

Updated version: Mireille Moisan-Sanscartier

Consultant in andragogy: Serge Vallières

Coordinator for the DGFD: Jean-Paul Groleau

Coordinator for the DFGA: Ronald Côté

Photocomposition and layout: Multitexte Plus

Translation: Consultation en éducation Zegray

Linguistic revision: Kay Flanagan and Leslie Macdonald

Translation of updated sections: Claudia de Fulviis

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UNITS

INTRODUCTION TO THE PROGRAM FLOWCHART

Welcome to the World of Mathematics!

This mathematics program has been developed for the adult students of the Adult Education Services of school boards and distance education. The learning activities have been designed for individualized learning. If you encounter difficulties, do not hesitate to consult your teacher or to telephone the resource person assigned to you. The following flowchart shows where this module fits into the overall program. It allows you to see how far you have progressed and how much you still have to do to achieve your vocational goal. There are several possible paths you can take, depending on your chosen goal.

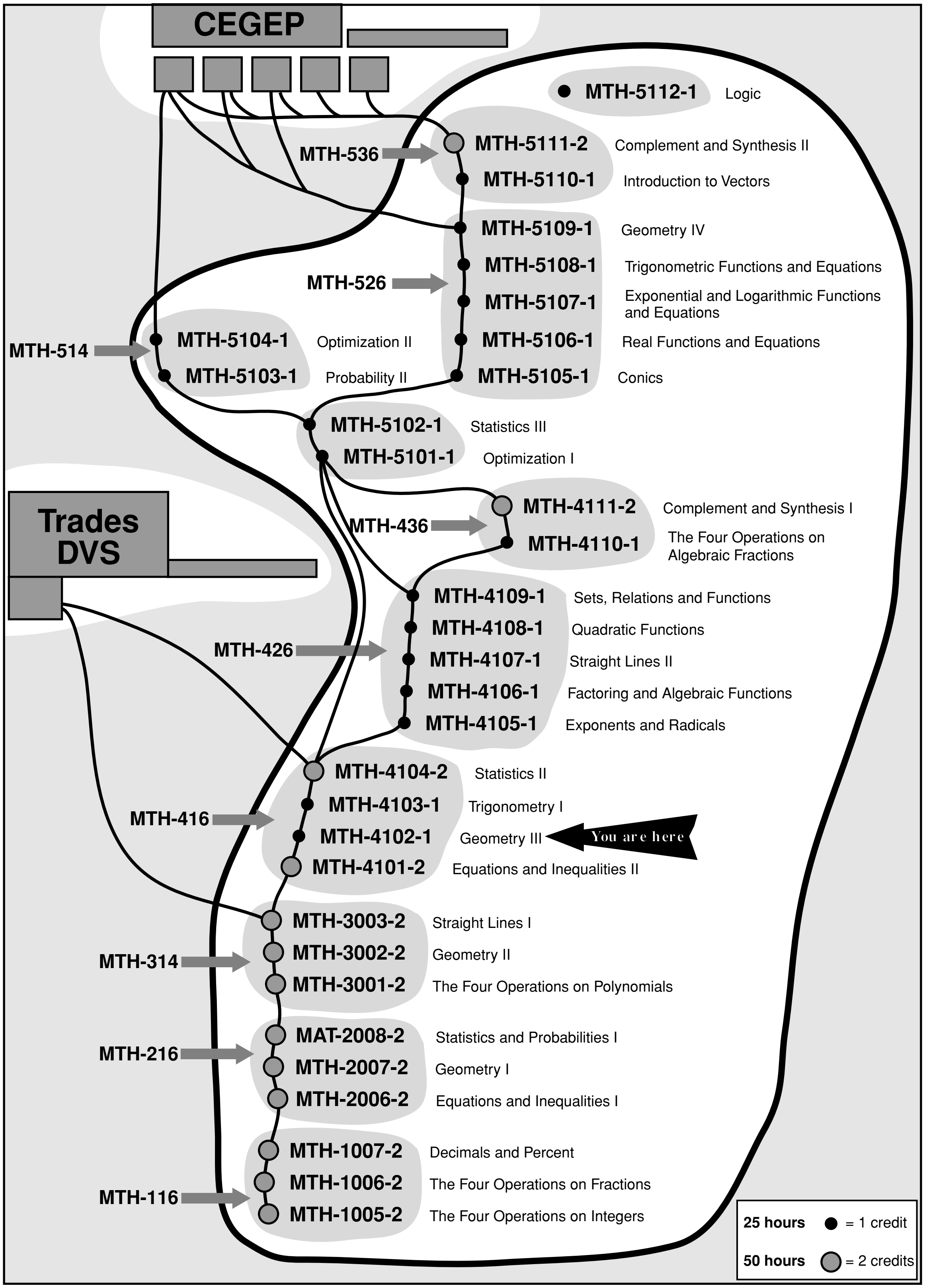
The first path consists of modules MTH-3003-2 (MTH-314) and MTH-4104-2 (MTH-416), and leads to a Diploma of Vocational Studies (DVS).

The second path consists of modules MTH-4109-1 (MTH-426), MTH-4111-2 (MTH-436) and MTH-5104-1 (MTH-514), and leads to a Secondary School Diploma (SSD), which allows you to enroll in certain Gegep-level programs that do not call for a knowledge of advanced mathematics.

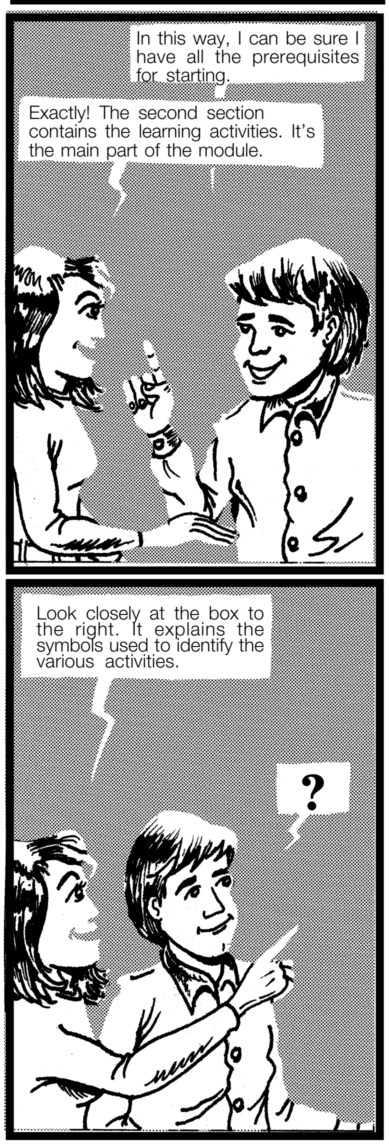
The third path consists of modules MTH-5109-1 (MTH-526) and MTH-5111-2 (MTH-536), and leads to Cegep programs that call for a solid knowledge of mathematics in addition to other abiliies.

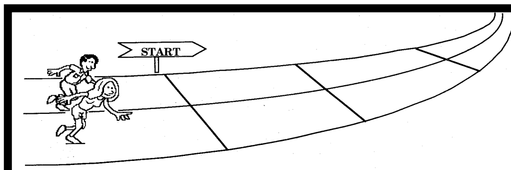
If this is your first contact with this mathematics program, consult the flowchart on the next page and then read the section “How to Use This Guide.” Otherwise, go directly to the section entitled “General Introduction.” Enjoy your work!

THE PROGRAM FLOWCHART



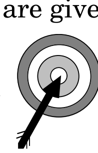
HOW TO USE THIS GUIDE



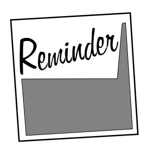


The starting line shows where the learning activities **begin**.

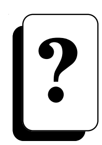
The little white question mark indicates the **questions** for which answers are given in the text.



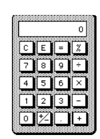
The target precedes the **objective** to be met.



The memo pad signals a **brief reminder** of concepts which you have already studied.

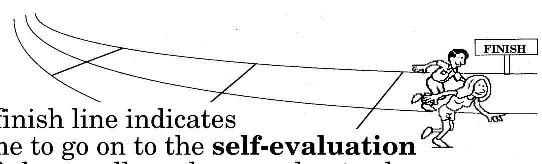


The boldface question mark indicates **practice exercices** which allow you to try out what you have just learned.



The **calculator** symbol reminds you that you will need to use your calculator.

The sheaf of wheat indicates a **review** designed to reinforce what you have just learned. A row of sheaves near the end of the module indicates the final review, which helps you to interrelate all the learning activities in the module.



Lastly, the finish line indicates that it is time to go on to the **self-evaluation test** to verify how well you have understood the learning activities.


There are also many fun things in this module. For example, when you see the drawing of a sage, it introduces a "Did you know that..."
A "Did you know that..."?
Yes, for example, short tidbits on the history of mathematics and fun puzzles. They are interesting and relieve tension at the same time.
Must I memorize what the sage says?
No, it's not part of the learning activity. It's just there to give you a breather.
It's the same for the "math whiz" pages, which are designed especially for those who love math.
They are so stimulating that even if you don't have to do them, you'll still want to.
And the whole module has been arranged to make learning easier.
For example. words in boldface italics appear in the glossary at the end of the module...
Great!
... statements in boxes are important points to remember, like definitions, formulas and rules. I'm telling you, the format makes everything much easier.
The third section contains the final review, which interrelates the different parts of the module.
There is also a self-evaluation test and answer key. They tell you if you're ready for the final evaluation.
Thanks, Monica, you've been a big help.
I'm glad! Now, I've got to run. See you!
Later ...
This is great! I never thought that I would like mathematics as much as this!

GENERAL INTRODUCTION

CONGRUENT FIGURES AND SIMILAR FIGURES

The properties of congruent figures and of similar figures are applied in science, photography, surveying, architecture, carpentry, machining and geography. These concepts are also very useful when we need to read or draw a scale diagram and to calculate distances using a road map. However, we often apply these properties automatically without even realizing it. The theory explained in this module will allow you to broaden your awareness and knowledge of them.

Two types of **geometric transformations** will be introduced first: **isome-tric transformations**, also called **isometries**, and **transformations of simili-tude**, also called **size transformations**. An isometric transformation changes a given geometric figure into another figure which has the same size and the same shape. These two figures are therefore congruent. The following transfor-mations are isometries: **translation**, **rotation** and **reflection**. On the other hand, a transformation of similitude changes a given geometric figure into another whose shape is the same, but whose size is different. These figures are said to be similar. In other words, a transformation of similitude produces an enlarged or reduced image of the original object. This is why it is also referred to as a size transformation.

You will see how to **draw a unique triangle**, given the measure of the angle contained between two given sides, the length of the side contained between two given angles, or the lengths of its three sides. Then, the properties of **congruent** and **similar** triangles will be introduced. That is, if two triangles have **a congruent angle contained between two congruent sides** (S-A-S prop-erty), **a congruent side contained between two congruent angles** (A-S-A property) or **three congruent sides** (S-S-S property), these triangles are congruent. When two triangles are similar, their **corresponding angles are congruent** and their **corresponding sides are proportional**. If you know that two triangles have **two congruent angles** (A-A property), **three corresponding proportional sides** (S-S-S property) or **a congruent angle contained between two corresponding proportional sides** (S-A-S property), you can conclude that these triangles are similar.

Once you are able to identify the corresponding sides of two **similar triangles**, you will learn how to **calculate the lengths of the missing sides** given the necessary measures. Sometimes you will be given the value of the ratio of similitude between the corresponding sides of these triangles. You will also deal with cases where this ratio is not known. Since these concepts of similarity also apply to **similar polygons**, you will then be able to determine the lengths of the missing sides of two similar polygons for which you are given sufficient information.

After this, **scale diagrams** or plans will be considered. The diagram or plan of an object is a reduced image which is similar to the actual object. The scale of the plan is the ratio of similitude which exists between these two figures. You will therefore be able to apply the concepts already learned to calculate actual measures given a plan whose scale is known. Conversely, you will learn how to determine the lengths of segments for the purpose of drawing the plan of an object to a given scale.

Lastly, you will use these newly acquired concepts to **solve problems** related to various fields of human activity such as photography, surveying, carpentry, machining and geography, which involve concepts of similarity and congruence of figures.

INTERMEDIATE AND TERMINAL OBJECTIVES OF THE MODULE

Module MTH-4102-1 consists of eight units and requires twenty-five hours of study distributed as follows. Each unit covers either an intermediate or a terminal objective. The terminal objectives appear in boldface.

Objectives	**Number of Hours***	**% (evaluation)**
1	4	10%
2	4	10%
3	5	20%
4	4	25%
5	2	5%
6 to **8**	5	30%

* One hour is allotted for the final evaluation.

1. Isometric Transformations: Translation, Rotation and Reflection

Among a set of illustrations representing isometric transformations of geometric figures, to recognize:

- **those which illustrate a translation**
- **those which illustrate a rotation**
- **those which illustrate a reflection**

and, using a ruler, set-square, compass or protractor, to draw the images of simple geometric figures under the following transformations:

- **a translation *t*, given the length and direction of its displacement**
- **a rotation *r*, given the location of its center of rotation and the measure of its angle of rotation**
- **a reflection *s*, given the location of its axis of reflection.**

2. Transformations of Similitude and Similar Figures

Using a ruler and set-square, to draw the image of a geometric figure under a size transformation *h*, given the location of the center of similitude (O) and the ratio of similitude (*k*). The value of *k* can be either positive or negative. Also, from among a set of illustrations representing geometric transformations, to recognize those which illustrate a size transformation *h*.

3. Congruent Triangles and Similar Triangles

Using a ruler, protractor and compass, to draw the unique triangle defined by one of the following groups of measures:

- **an angle and the two sides which form this angle**
- **two angles and the side contained between these angles**
- **the three sides**

and, given the measures of some of the angles and some of the sides of two triangles, to determine whether the triangles are congruent or similar by applying the properties of congruent triangles and those of similar triangles. Reasons must be given for each conclusion.

4. Calculating the Lengths of Sides of Two Similar Triangles

To calculate the length of one or more sides of one of two similar triangles, given:

- **the length of one or more corresponding sides of one of the triangles**
- **the value of the ratio of similitude *k* or the lengths required to calculate *k*.**

The steps in the solution must be described.

5. Calculating the Lengths of Sides of Two Similar Polygons

To calculate the length of one or more sides of one of two similar polygons, given:

- **the length of one or more corresponding sides in one of the polygons**
- **the value of the ratio of similitude *k* or the lengths required to calculate *k*.**

The polygons have a maximum of eight sides. The steps in the solution must be described.

6. Calculating Actual Dimensions, Given a Scale Diagram

To apply the properties of similar figures to solve word problems which require the calculation of the actual measure of certain lengths, given a scale diagram illustrating an everyday situation.

7. Drawing a Scale Diagram, Given Actual Dimensions

To apply the properties of similar figures to solve problems which involve drawing a scale diagram using a ruler and set-square, given a drawing that illustrates an everyday situation and the numerical value of the scale for the diagram that is required. Only angles of 90° will be represented.

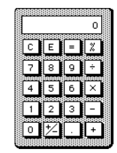
8. Solving Problems from Various Areas of Human Endeavor Involving Concepts of Similarity or of Congruence in Geometric Figures

To apply the properties of congruent figures and those of similar figures to solve problems which require the calculation of the measures of angles and of actual lengths, given the dimensions on a scale diagram, or problems which require the calculation of measures to be used in a scale diagram, given actual dimensions. The use of a ruler, set-square and protractor is required. A drawing that illustrates situations borrowed from everyday life will be given. The steps in the solution must be described.

DIAGNOSTIC TEST ON THE PREREQUISITES

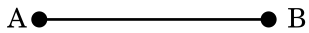
Instructions

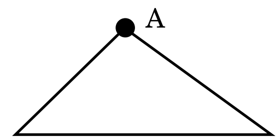
1. Answer as many questions as you can.
2. To answer these questions, you should have the following instruments on hand: a ruler graduated in centimetres and in millimetres, a protractor and a calculator.

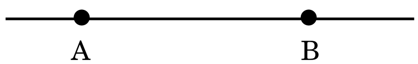


3. Write your answers on the test paper.
4. Do not waste time. If you cannot answer a question, go on to the next one immediately.
5. When you have answered as many questions as you can, correct your answers, using the answer key which follows the diagnostic test.
6. To be considered correct, your answers must be identical to those in the key. For example, if you are asked to describe the steps involved in solving a problem, your answer must contain all the steps.
7. Transcribe your results onto the chart which follows the answer key. It gives an analysis of the diagnostic test results.
8. Do only the review activities which are suggested for each of your incorrect answers.
9. If all your answers are correct, you may begin working on this module.

1. Given the figures illustrated below,

1) 

2) 

3) 

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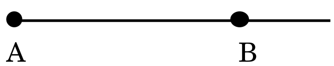
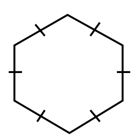
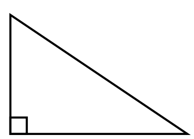
5) 

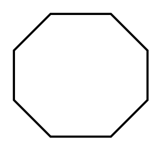
write the number corresponding to each figure beside its correct name:

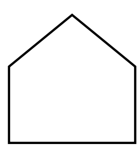
a) point A: ....
b) line AB: ....
c) line segment AB: ....
d) ray AB: ....
e) vertex A of a figure: ....

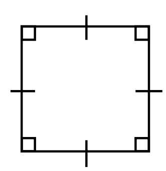
2. Given the geometric figures below,

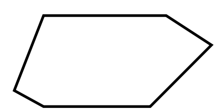
1) 

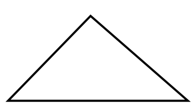
2) 

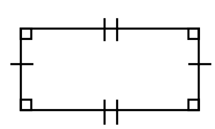
3) 

4) 

5) 

6) 

7) 

8) 

write the number corresponding to each figure beside its correct name:

a) pentagon: ....
b) square: ....
c) triangle: ....
d) rectangle: ....
e) right triangle: ....
f) octagon: ....
g) hexagon: ....
h) regular hexagon: ....

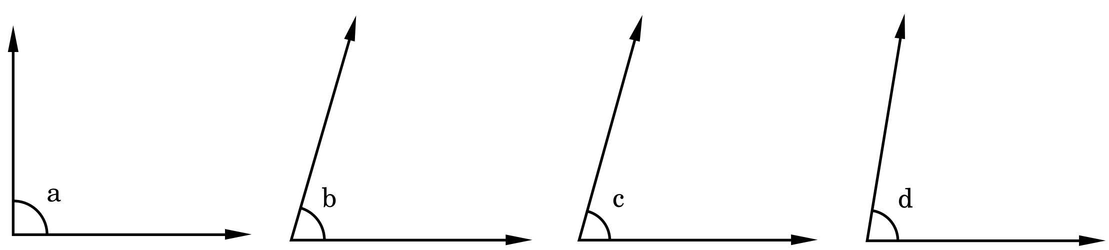
3. Read the following definitions.

   1) Closed figure with four sides.
   2) Figure formed by two rays with a common endpoint.
   3) Closed figure formed by line segments.
   4) Lines which never meet; the distance between them is always the same.
   5) Sum of the lengths of the sides of a closed figure.
   6) Lines intersecting at a 90° angle.
   7) Closed figure with four sides whose opposite sides are parallel.

   Write the number corresponding to each definition beside the correct term below:

   a) perimeter: ....
   b) polygon: ....
   c) quadrilateral: ....
   d) parallelogram: ....
   e) angle: ....
   f) parallel lines: ....
   g) perpendicular lines: ....

4. Examine the angles below.



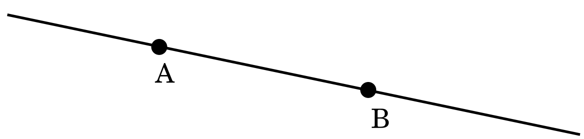
$m\angle a = 90°$ $m\angle b = 75°$ $m\angle c = 75°$ $m\angle d = 80°$

   a) The right angle is ....
   b) The pair of congruent angles is ....

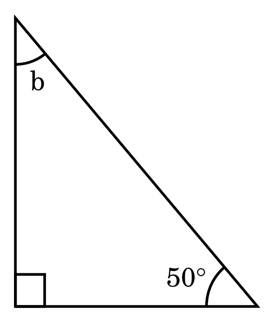
5. a) Using a ruler, draw a line segment AB 3.6 cm long.
   N.B. A precision of ±1 mm is required.

b) Using a protractor, draw an angle RST of 60°.
N.B. The drawing must be precise to within ±2°.

c) Using a set-square, draw a line CD perpendicular to line AB illustrated below.



6. Determine the measure of angle b in the adjacent triangle.
N.B. Do not use a protractor.



7. a) Given that a pencil measures 15 cm and that a match measures 3 cm, what is the ratio between the length of the match and that of the pencil? Simplify this ratio to lowest terms.

b) The perimeter of a square is 16 cm while that of a rectangle is 4 cm. What is the ratio between the perimeter of the square and that of the rectangle? Simplify this ratio to lowest terms.

8. Determine the value of x in the following proportions. A detailed solution is required.

a) $\frac{1}{7} = \frac{x}{147}$

b) $\frac{3}{5} = \frac{81}{x}$

9. Convert the following measures to the unit required.

a) 4 000 cm is equivalent to .... m

b) 2.5 km is equivalent to .... cm

c) 1 000 000 cm is equivalent to ...... km

d) 3 m is equivalent to .... cm

10. Solve the following problems and give the steps followed in calculating the answers.

    a) Renélee and Lucy collect erasers. Renélee has 25 and Lucy has 33. Lucy has decided to collect several more and to give them to Renélee for her birthday: each time she buys 5 erasers, she keeps 3 for Renélee. By the time Lucy has bought 45 erasers, how many will she have to give to Renélee?

    b) Francis has a monthly income of $1 250. He notes that his rent costs a fifth of the money he earns. Next year his employer will give him a 10% raise in salary: Francis' monthly income will then be $1 375. If he sets aside the same fraction of his salary for his rent, what amount of rent will Francis then pay each month?

ANSWER KEY FOR THE DIAGNOSTIC TEST ON THE PREREQUISITES

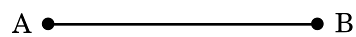
1. a) 4 b) 3 c) 1 d) 5 e) 2

2. a) 4 b) 5 c) 7 d) 8 e) 2 f) 3
   g) 6 h) 1

3. a) 5 b) 3 c) 1 d) 7 e) 2 f) 4
   g) 6

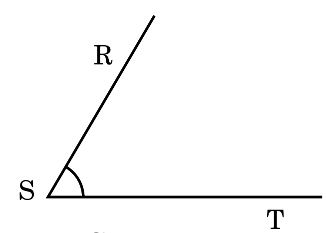
4. a) ∠a b) ∠b and ∠c

5. a)



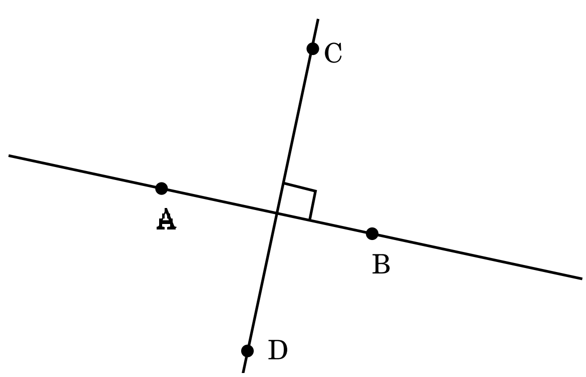
$m\overline{AB} = 3.6$ cm

b)



m∠RST = 60°

c)



N.B. The perpendicular line could cut the original line at any point as long as it cuts it at an angle of 90°.

6. m∠b = 40° since 180° – (90° + 50°) = 40°. The sum of the measures of the three angles of a triangle is 180°.

7. a) $\frac{3\text{ cm}}{15\text{ cm}} = \frac{1}{5}$ b) $\frac{16\text{ cm}}{4\text{ cm}} = \frac{4}{1} = 4$

8. a) $\frac{1}{7} = \frac{x}{147}$

$147 = 7x$

$\frac{147}{7} = x$

$x = 21$

b) $\frac{3}{5} = \frac{81}{x}$

$3x = 405$

$x = \frac{405}{3}$

$x = 135$

9. a) 40 m b) 250 000 cm c) 10 km d) 300 cm

10. a) You know that 3 erasers out of 5 are intended for Renélee. You can therefore write the following proportion:

$$\frac{3}{5} = \frac{x}{45}$$

$$135 = 5x$$

$$x = 27$$

N.B. You can also say that $\frac{3}{5}$ of the erasers are for Renélee, from which $x = \frac{3}{5} \times 45 = 27$.

- Lucy will have 27 erasers for Renélee.

b) His monthly rent will be equal to $\frac{1}{5}$ of his monthly salary:

$$\frac{1}{5} = \frac{x}{\$1\,375}$$

$$\$1\,375 = 5x$$

$$x = \frac{\$1\,375}{5}$$

$$x = \$275$$

N.B. You could also write: $x = \frac{1}{5} \times \$1\,375 = \$275$.

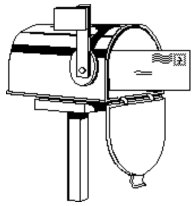
- Francis will pay $275 per month in rent.

ANALYSIS OF THE DIAGNOSTIC TEST RESULTS

Question	Answer		Review		Before Going to Unit(s)
	Correct	Incorrect	Section	Page	
1.			10.2	10.18	1
2.			10.3	10.18	1
3.			10.4	10.18	1
4.			10.5	10.18	1
5. a)			10.6	10.23	1
b)			10.6	10.26	1
c)			10.6	10.29	1
6.			10.7	10.33	3
7.			10.7	10.36	3
8.			10.7	10.38	4
9.			10.7	10.41	6
10.			10.7	10.4	1

- If all your answers are **correct**, you may begin working on this module.
- For each **incorrect** answer, find the related section listed in the **Review** column. Do the review activities for that section before beginning the units listed in the right-hand column under the heading **Before Going to Unit(s)**.

Answer Key



INFORMATION FOR DISTANCE EDUCATION STUDENTS

You now have the learning material for MTH-4102-1 together with the homework assignments. Enclosed with this material is a letter of introduction from your tutor indicating the various ways in which you can communicate with him or her (e.g. by letter, telephone) as well as the times when he or she is available. Your tutor will correct your work and help you with your studies. Do not hesitate to make use of his or her services if you have any questions.

DEVELOPING EFFECTIVE STUDY HABITS

Distance education is a process which offers considerable flexibility, but which also requires active involvement on your part. It demands regular study and sustained effort. Efficient study habits will simplify your task. To ensure effective and continuous progress in your studies, it is strongly recommended that you:

- draw up a study timetable that takes your working habits into account and is compatible with your leisure time and other activities;
- develop a habit of regular and concentrated study.

The following guidelines concerning the theory, examples, exercises and assignments are designed to help you succeed in this mathematics course.

Theory

To make sure you thoroughly grasp the theoretical concepts:

1. Read the lesson carefully and underline the important points.

2. Memorize the definitions, formulas and procedures used to solve a given problem, since this will make the lesson much easier to understand.

3. At the end of an assignment, make a note of any points that you do not understand. Your tutor will then be able to give you pertinent explanations.

4. Try to continue studying even if you run into a particular problem. However, if a major difficulty hinders your learning, ask for explanations before sending in your assignment. Contact your tutor, using the procedure outlined in his or her letter of introduction.

Examples

The examples given throughout the course are an application of the theory you are studying. They illustrate the steps involved in doing the exercises. Carefully study the solutions given in the examples and redo them yourself before starting the exercises.

Exercises

The exercises in each unit are generally modelled on the examples provided. Here are a few suggestions to help you complete these exercises.

1. Write up your solutions, using the examples in the unit as models. It is important not to refer to the answer key found on the coloured pages at the end of the module until you have completed the exercises.

2. Compare your solutions with those in the answer key only after having done all the exercises. **Careful!** Examine the steps in your solution carefully even if your answers are correct.

3. If you find a mistake in your answer or your solution, review the concepts that you did not understand, as well as the pertinent examples. Then, redo the exercise.

4. Make sure you have successfully completed all the exercises in a unit before moving on to the next one.

Homework Assignments

Module MTH-4102-1 contains three assignments. The first page of each assignment indicates the units to which the questions refer. The assignments are designed to evaluate how well you have understood the material studied. They also provide a means of communicating with your tutor.

When you have understood the material and have successfully done the pertinent exercises, do the corresponding assignment immediately. Here are a few suggestions.

1. Do a rough draft first and then, if necessary, revise your solutions before submitting a clean copy of your answer.

2. Copy out your final answers or solutions in the blank spaces of the document to be sent to your tutor. It is preferable to use a pencil.

3. Include a clear and detailed solution with the answer if the problem involves several steps.

4. Mail only one homework assignment at a time. After correcting the assignment, your tutor will return it to you.

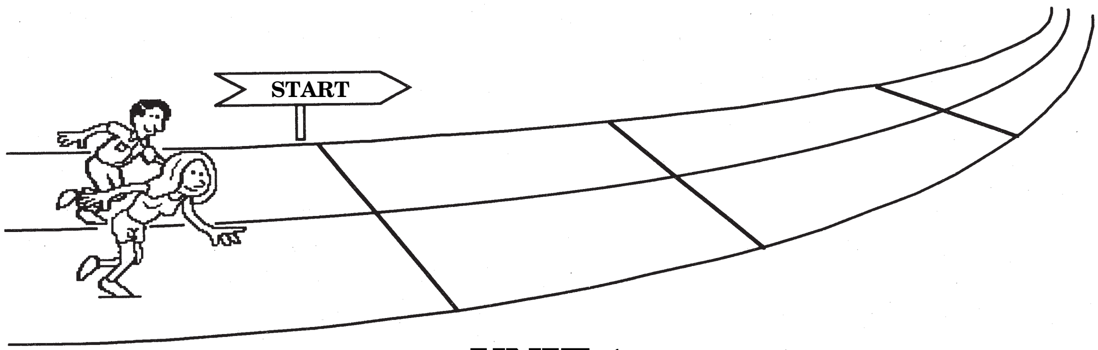
In the section "Student's Questions", write any questions which you may wish to have answered by your tutor. He or she will give you advice and guide you in your studies, if necessary.

In this course

Assignment 1 is based on units 1 to 3.
Assignment 2 is based on units 4 to 8.
Assignment 3 is based on units 1 to 8.

CERTIFICATION

When you have completed all the work, and provided you have maintained an average of at least 60%, you will be eligible to write the examination for this course.



UNIT 1

ISOMETRIC TRANSFORMATIONS: TRANSLATION, ROTATION AND REFLECTION

1.1 SETTING THE CONTEXT

Denis' Room

For a change, Denis periodically rearranges the furniture in his room. This time he wants to put a mirror on the wall and to move his armchair. The illustration below shows the changes.

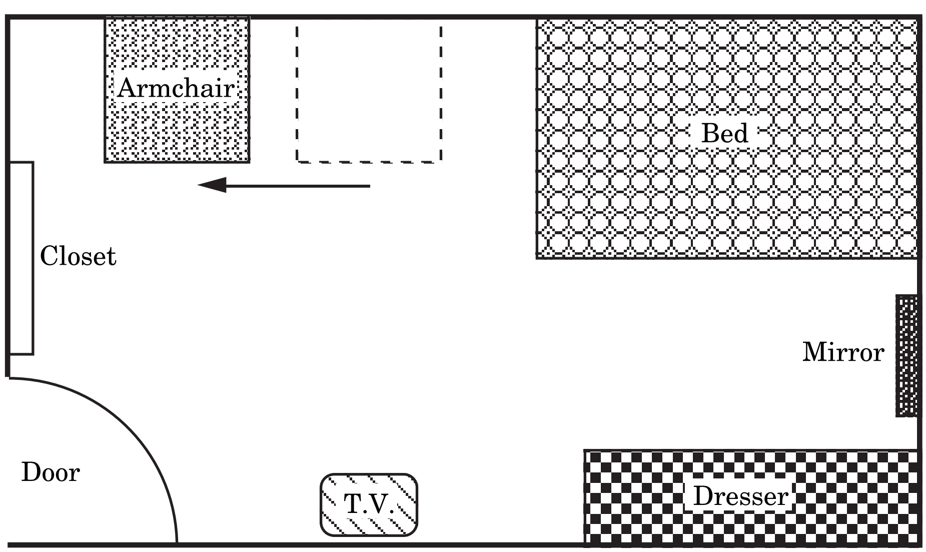


Fig. 1.1 Plan of Denis' room

When he moved his armchair along the wall, Denis performed a ***geometric transformation*** which is called a ***translation***. An automobile moving in a straight line or the sliding of a drawer as it is being opened or closed are other examples of translations.

That evening Denis watched television. To make himself comfortable, he had to move his armchair as shown in the following illustration.

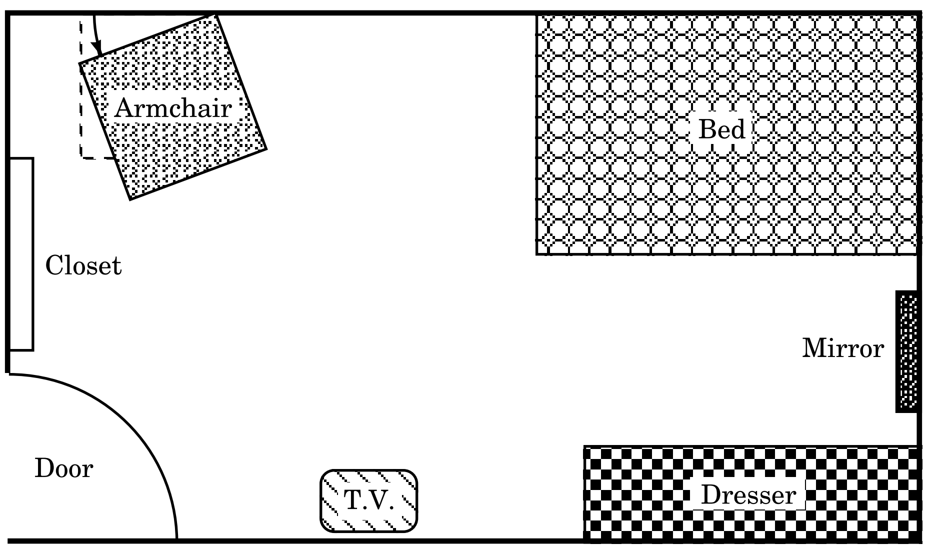


Fig. 1.2 Plan of Denis' room after
he moved the armchair a second time

This type of transformation is called a ***rotation***. The movement of the hands of a clock and the back-and-forth motion of a swing are other examples of rotations.

Before going to bed, Denis looked at himself in the mirror. He noticed that objects that were really on his right appeared as if they were on his left in the mirror. This is an example of another type of transformation which is called a ***reflection***.

To reach the objective of this unit, you should be able to identify the type of isometric geometric transformation (translation, rotation or reflection) which changes one given figure into another given figure. You should also be able to draw the figure obtained when a given figure undergoes a given transformation (translation, rotation or reflection).

The transformations which have just been described are examples of ***isometries***.

> An **isometry** is a geometric transformation which preserves length. Translations, rotations and reflection are isometries

In other words, an object which has undergone a translation, a rotation or a reflection has the same dimensions as the original object.

Each of these isometries will now be examined in detail.

1.1.1 Translation

> A **translation** is the movement of a given object in a constant direction in a plane.

Translations are represented by a directed arrow showing the direction and magnitude of the displacement. Do you remember Figure 1.1, which shows a plan of Denis' room after the armchair was moved the first time? The armchair was moved to the left in a constant direction, ***parallel*** to the wall. A translation involves drawing a line parallel to a given line.

Reminder

Two lines are parallel if the distance between them is constant.

To draw a line parallel to line *l*, using a ruler and *set-square*:

1. Align one of the legs of the ***right angle*** of the set-square with line *l*.
2. Place the ruler against the other leg of the right angle of the set-square as illustrated in Figure 1.4.
3. Slide the set-square along the ruler until the desired point is reached being careful to keep the ruler still.
4. Draw the line *l*' parallel to line *l* through the selected point P.

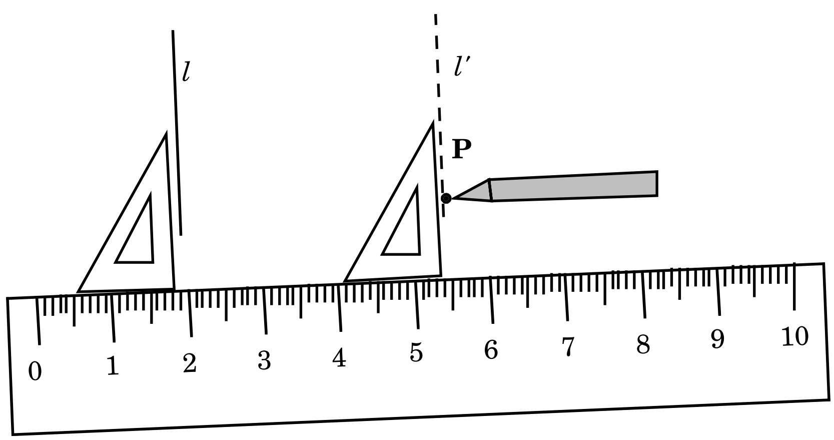


Fig. 1.3 Drawing a line *l*' parallel to line *l*

You are now ready to draw a translation.

Example 1

Consider ΔABC shown on the right. Draw the image of this triangle under translation *t*.

1. From A, draw an arrow the **same length** as, in the **same direction** as and **parallel to *t***, and call the tip of this A'. Point A has been moved to A'.
2. Repeat this operation starting from points B and C. Point B has been moved to B' and point C has been moved to C'.
3. Join points A', B' and C'. You obtain ΔA'B'C', which is the ***image*** of ΔABC under translation *t*.

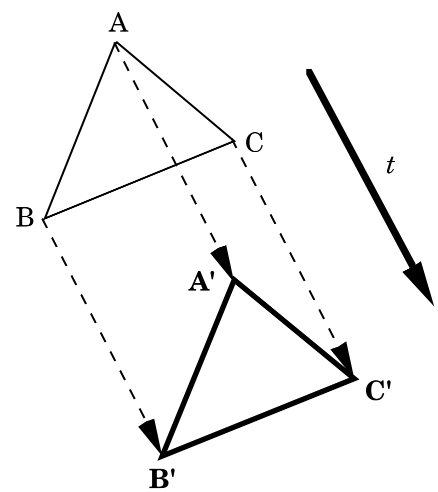


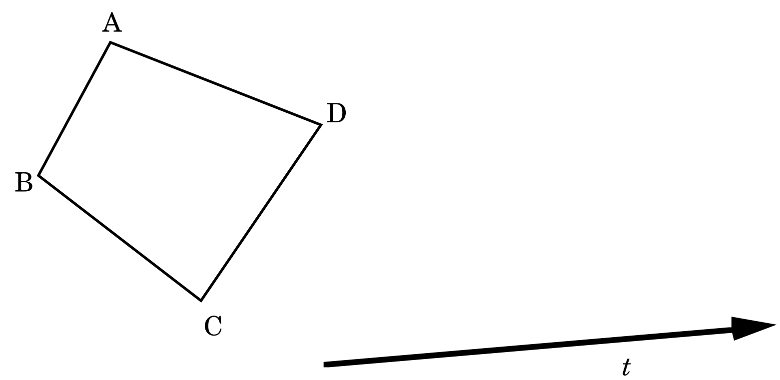
Fig. 1.4 Triangle ABC under translation *t*

N.B. A' is read A prime. B' is read B prime. C' is read C prime.

Remarks

1. The arrows all point in the same direction and they are ***parallel***.
2. All the displacements are of the same length *t*.
3. Note that the image of point A under translation *t* is t(A) or A'. The same is true for every other point in the figure.
   N.B. *t*(A) is read *t* of A
4. No point remains in a fixed position under a translation.

Perform translation *t* on ***quadrilateral*** ABCD as illustrated below.



1. If you drew, starting at point A, an arrow the **same length as**, in the **same direction** as and **parallel to *t***.

2. If you repeated this operation starting at each of the other ***vertices*** of the quadrilateral, that is, points B, C and D.

3. If you joined points A' B' C' and D', you obtained quadrilateral A'B'C'D' as illustrated at the Figure 1.5 below.

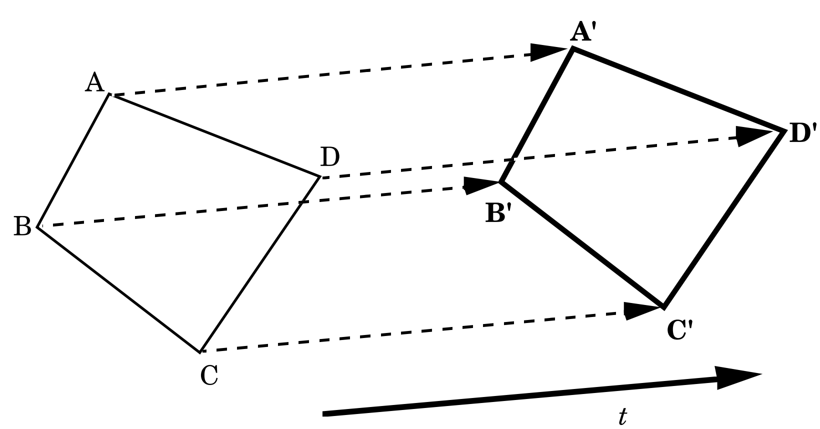


Fig. 1.5 Quadrilateral ABCD undergoing translation ***t***

We say that quadrilateral A'B'C'D' is the **image** of quadrilateral ABCD under translation ***t***.

Properties of a translation

Any translation:

1. Transforms a line into another line parallel to it.
2. Preserves the lengths of ***line segments***.
3. Preserves angle measures.
4. Preserves the order of points, that is, if a point is the midpoint of one side of a figure, its image will also be the midpoint of the corresponding side of the image of this figure.
5. Preserves the parallelism of lines, that is, if two sides of a figure are parallel, the corresponding sides will also be parallel in the image of this figure.
6. Preserves the ***perpendicularity*** of lines, that is, if two sides of a figure are perpendicular, the corresponding sides in the image of this figure will also be perpendicular.
7. Preserves the ***ratio*** of lengths.

These properties are said to be the ***invariants*** of a translation.

1.1.2 Rotation

A **rotation** is the change in a figure's position determined by an angle of rotation in a given direction about a point called the ***centre of rotation***.

There exist two directions of rotation:

1. ***clockwise*** direction, or the direction in which the hands of a watch rotate, also called negative direction;

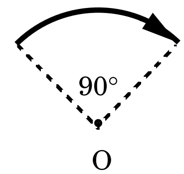


Fig. 1.6 Clockwise rotation of 90° about point O

2. ***counterclockwise*** direction, or the opposite direction to that in which the hands of a watch rotate, also called positive direction.

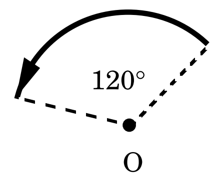
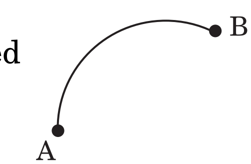


Fig. 1.7 Counterclockwise rotation of 120° about point O

Rotations are represented by an arc arrow which indicates the direction of the rotation and the size of the angle of rotation. The point about which the arc is drawn is called the center of rotation.

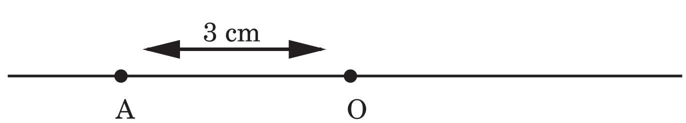
An arc of a circle is the part of a circle contained between two points.



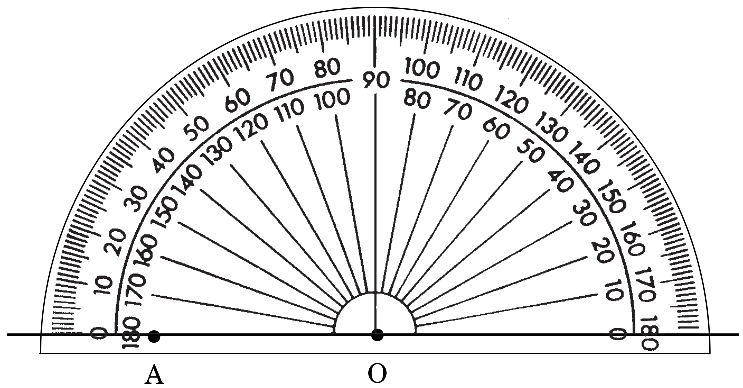
The procedure for drawing arc AB of a circle using a protractor and a compass is given below.

To draw an arc measuring 100° for which the compass opening is 3 cm:

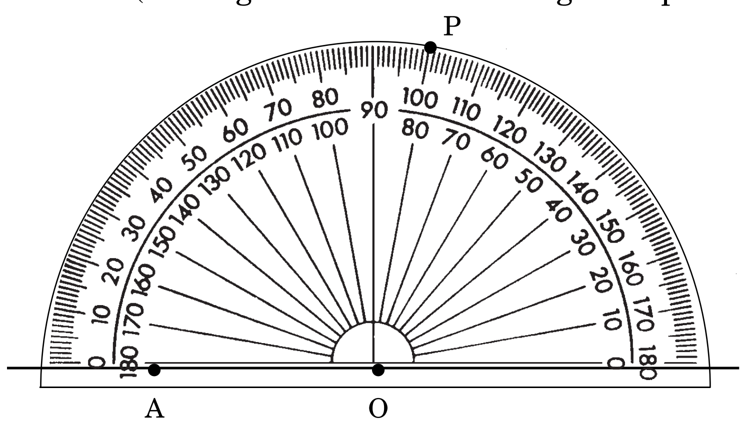
1. Draw a line AO with points A and O located 3 cm apart;



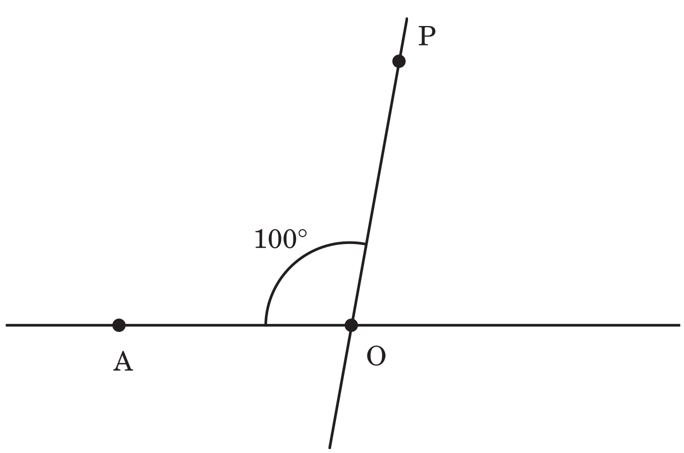
2. Place the straight edge of the protractor on line AO, making sure that the centre of the protractor is aligned with point O;



3. Mark off point P flush with the 100° division on the protractor (the degrees are read starting from point A);

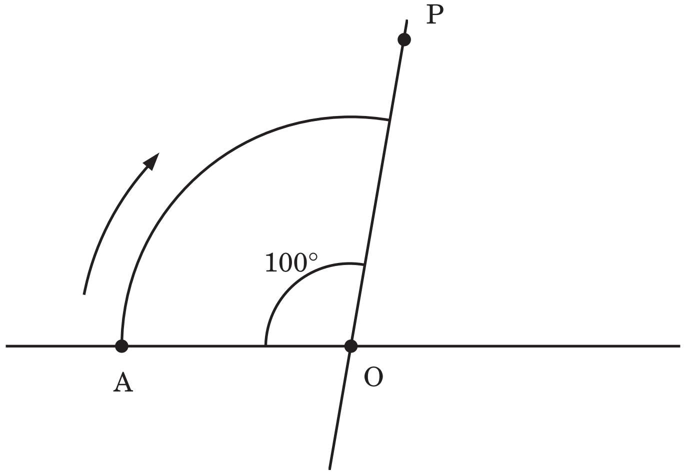


4. Remove the protractor and draw line OP;

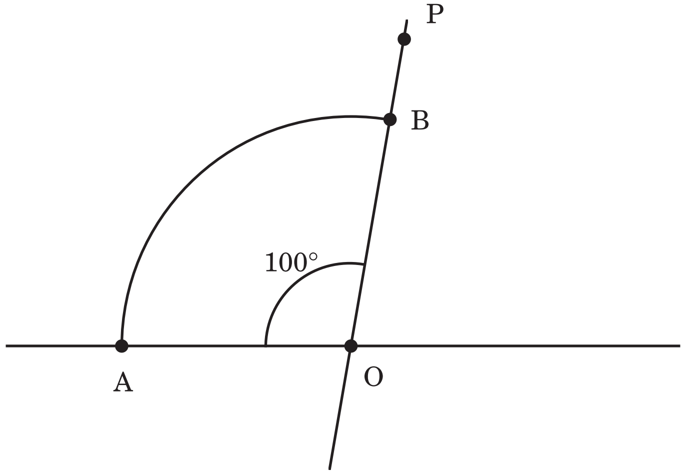


5. Place the pinpoint of the compass at point O and adjust the opening of the compass so that the pencil is on point A;

6. From point A, rotate the pencil about point O until this line cuts line OP;



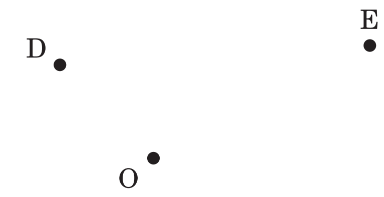
7. Mark off point B where the arc intersects line OP. The result is arc AB.



N.B. The arc should not extend beyond points A and B, since these two points define the arc.

You are now able to perform a rotation *r* from a given point to another point.

Example 2



Given points D and E on the right. Draw the image of these points by performing a clockwise rotation *r* through 90°.

1° Draw the dotted line segments OD and OE.

2° Draw counterclockwise arcs DD' and EE' of 90° using a protractor and a compass (m∠DOD' = 90° and m∠EOE' = 90°).

3° Points D' and E' are respectively the images of point D and E under rotation *r* through 90°.

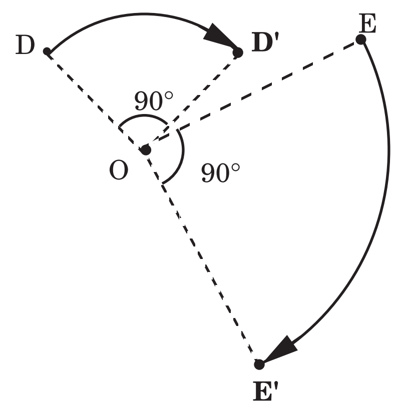
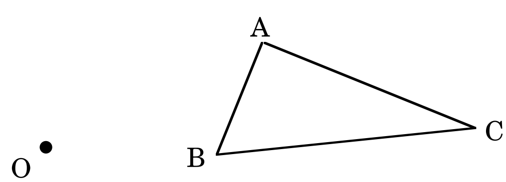


Fig. 1.8 Rotation *r* of points D and E about point O

? Perform a counterclockwise rotation *r* of ΔABC through 120° about point O.



You have obtained triangle A'B'C' in the Figure 1.9 below if you have:

1. Drawn the dotted line segment OA.
2. Drawn a counterclockwise arc AA' of 120° using a protractor and a compass (m∠AOA' = 120°).
3. Repeated this operation for points B and C, which are thus transformed to points B' and C' respectively (m∠BOB' = 120° and m∠COC' = 120°).
4. Joined points A', B' and C' to form triangle A'B'C'.

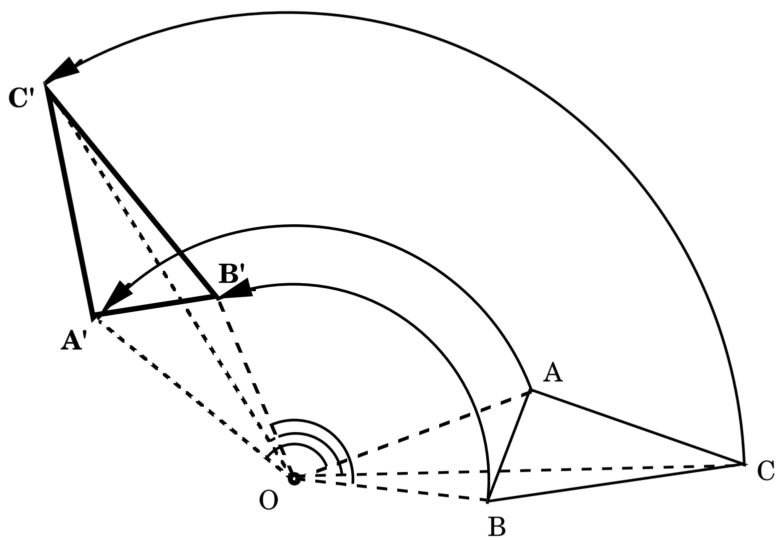


Fig. 1.9 Rotation *r* of triangle ABC about point O

Triangle A'B'C' is said to be the image of triangle ABC under rotation *r*; A' is the image of A, and points B' and C' are the respective images of B and C.

Remarks

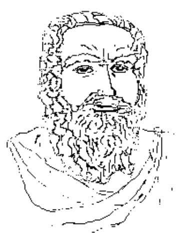
1. The arc arrows all point in the same direction: clockwise or counterclockwise.
2. All the arcs correspond to the same angle of rotation.
3. The image of point A under rotation *r* is written *r*(A) or A'; the image of every point in the geometric figure is represented in a similar way.
4. The only fixed point is the center of rotation.

Properties of a rotation

Any rotation:

1. Transforms a line into another line.
2. Preserves the lengths of line segments.
3. Preserves angle measures of geometric figures.
4. Preserves the order of points.
5. Preserves the parallelism of lines.
6. Preserves the perpendicularity of lines.
7. Preserves the ratio of lengths.

These properties of a rotation are said to be its **invariants**.



Did you know that

the long and short hands of a clock form right angles several times each day?

How many times does this occur in a 24-hour period?

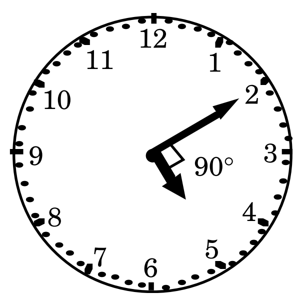


Fig. 1.10 Time at which the hands of a clock form a right angle

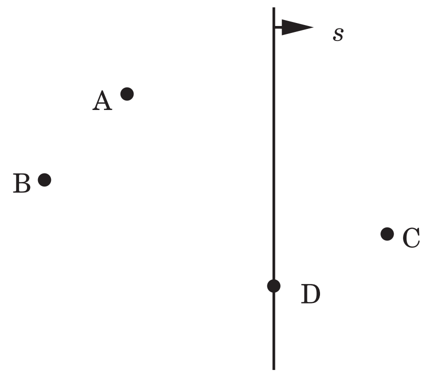
Answer given in the answer key.

1.1.3 Reflection

A **reflection** is a complete turnabout with respect to a line called the ***line of reflection***.

Example 3

Given the points A, B, C and D to reflect in the line of reflection *s*:



1. From point A, drop a dotted line perpendicular to the line of reflection *s* and extend it an equal distance on the other side. It goes without saying that the line of reflection passes through the midpoint of segment AA'.
2. Repeat this operation for points B and C, which are thus transformed to B' and C' respectively.
3. Since the distance from point D to the line of reflection is zero, the image D' of point D coincides with D itself.

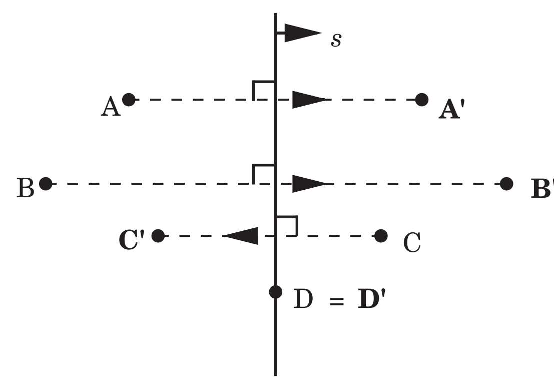
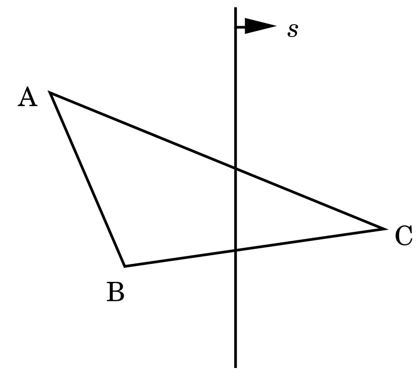


Fig. 1.11 Points A, B, C and D under reflection *s*

? Reflect triangle ABC in line *s*.



To obtain triangle A'B'C' as illustrated at Figure 1.12 below, you must have:

1. Dropped perpendiculars to the line of reflection *s* from the vertices A, B and C of the triangle.
2. Extended these perpendiculars an equal distance to A', B' and C'.
3. Joined A', B' and C' to form triangle A'B'C', which becomes the image of triangle ABC under reflection *s*.

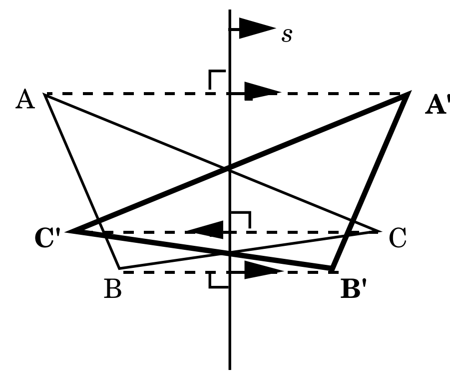


Fig. 1.12 Triangle ABC under reflection *s*

Remarks

1. The line of reflection is a ***line of symmetry*** (a line with respect to which points are symmetric in pairs).
2. The image of a point A under a reflection in a line *s* is written ***s*(A)** or **A'**.
3. When a figure is rotated a half-turn about a line, a reflection is obtained.
4. Points located on the line of reflection are fixed points, that is, points whose images coincide with these same points.

Properties of a reflection

Any reflection:

1. Transforms a line into another line.
2. Preserves the lengths of line segments.
3. Preserves the angle measures of geometric figures.
4. Preserves the order of points.
5. Preserves parallelism.
6. Preserves perpendicularity.
7. Preserves the ratio of lengths.

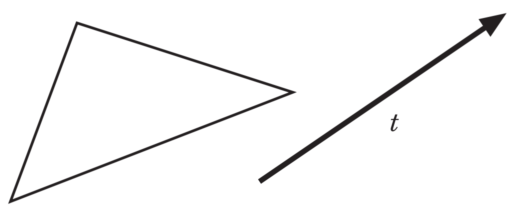
These properties of a reflection are said to be its **invariants**.

Are you ready to put this knowledge to work? If so, go on to the following activity. If not, reread the theory presented in this unit carefully.

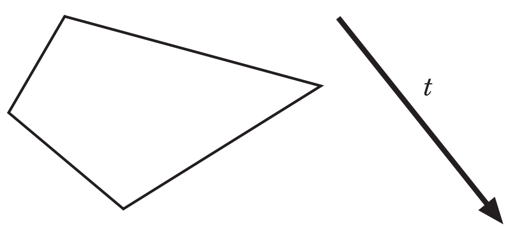
1.2 PRACTICE EXERCISES

1. Draw the image of each of the following figures under the translation represented.

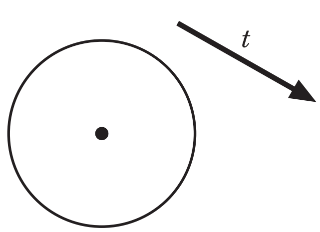
a)



b)

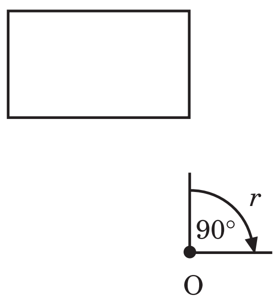


c)

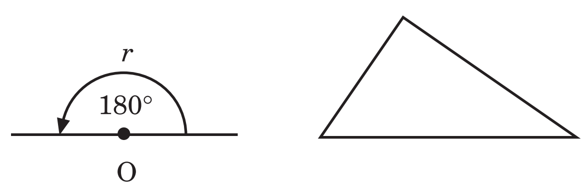


2. Draw the image of each of the following figures under the rotation represented.

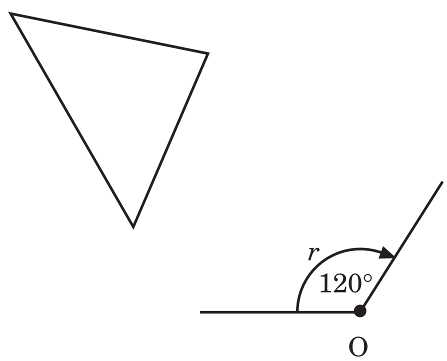
a)



b)

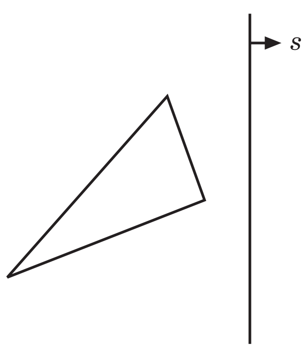


c)

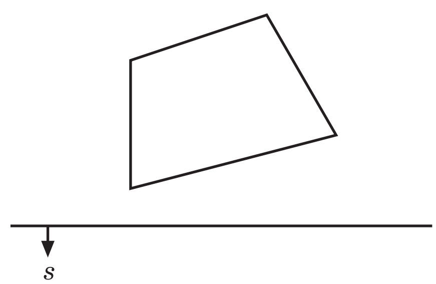


3. Draw the image of each of the following figures under the reflection represented.

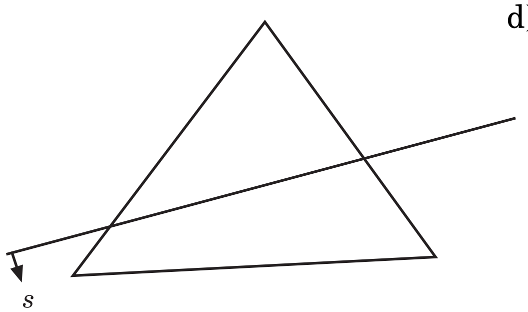
a)



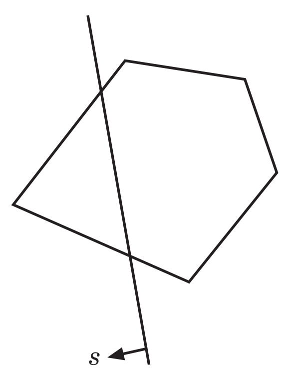
b)



c)



d)

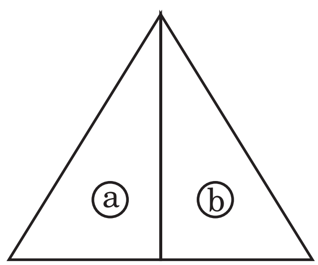


4. Identify the isometry which transforms figure ⓐ into figure ⓑ.
N.B. In some cases, there may be two answers.

a)

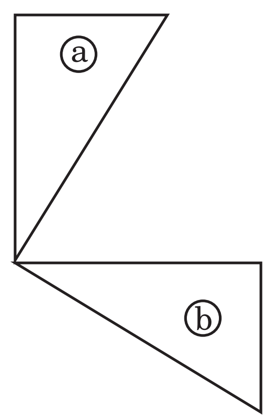
....

b)



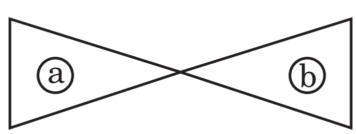
....

c)



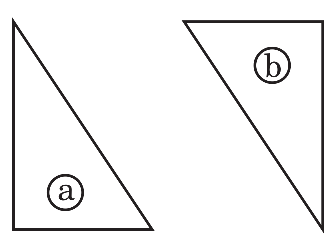
....

d)



....

e)



....

1.3 SUMMARY ACTIVITY

Refer to the theory presented in this unit to do this summary activity.

1. Match the following five terms with their respective definitions: isometry, rotation, translation, invariant, reflection.

   a) Quantity which does not change: ....

   b) Geometric transformation which consists in the turnabout of a figure with respect to a line: ....

   c) Geometric transformation which preserves length: ....

   d) Geometric transformation which consists in a displacement in a constant direction in a plane: ....

   e) Geometric transformation which consists in displacing a figure about a point: ....

2. Complete the following sentences:

   a) The point about which a rotation is performed is called ....

   b) The line with respect to which a reflection is performed is called ....

   c) The direction of rotation of the hands of a watch is called ....

3. Name the 7 properties which are the invariants for the 3 isometries (translation, rotation and reflection) presented in this unit.

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4. Which is the only isometry to have the property of transforming a line into another line **parallel to it**?

....

1.4 THE MATH WHIZ PAGE

1 Rotation = 2 Reflections!

To draw the image of triangle ABC under a rotation *r* of 120° in a clockwise direction about point O, we follow the method described in this unit and obtain triangle A'B'C'.

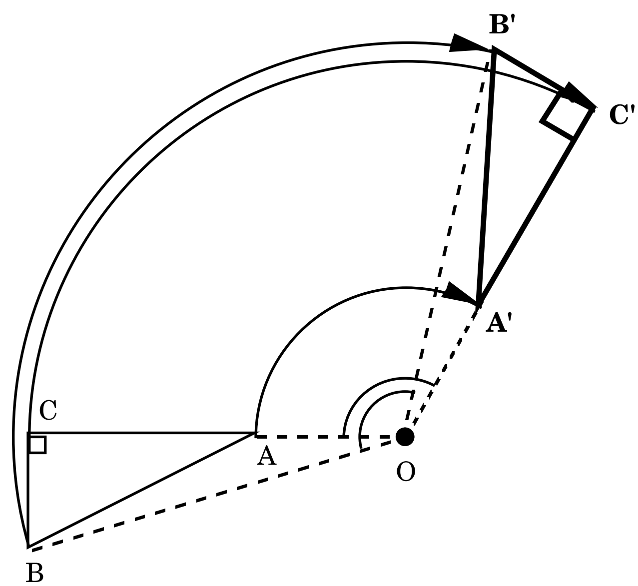

Fig. 1.13 Rotation *r* of triangle ABC about point O

However, the same result would have been obtained if two consecutive reflections had been performed. Look carefully at Figure 1.14 where triangle A''B''C'' represents the intermediate step that must be done to obtain image A'B'C' of triangle ABC.

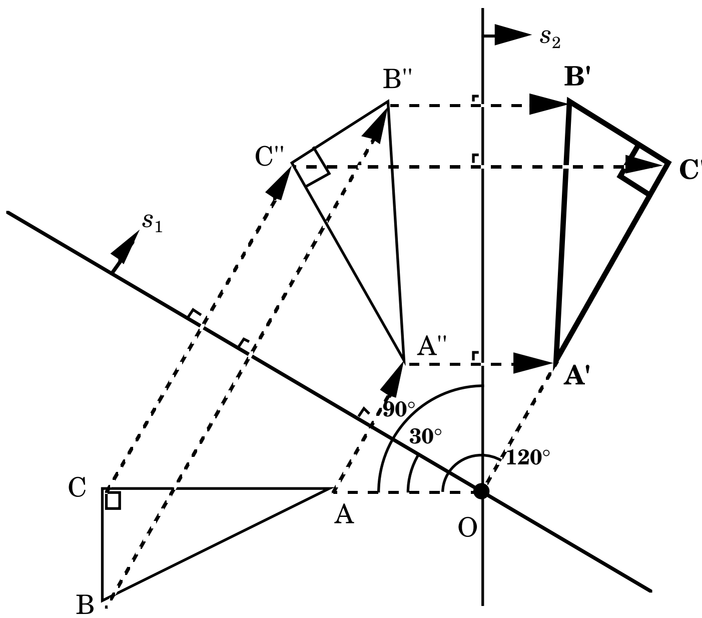

Fig. 1.14 Two consecutive reflections s_1 and s_2 of triangle ABC, which correspond to the rotation r about point O

To obtain the appropriate lines of reflection, we proceed as follows:

1. Draw a dotted line segment from point O to one of the vertices of the figure, for example, point A.

2. Draw line s_1 in such a way that it forms an angle whose measure is equal **to $\frac{1}{4}$ of the measure of the angle of rotation** with respect to segment OA. In the example, the measure of this angle is $\frac{1}{4} \times 120° = 30°$. Note that this angle is measured in a clockwise direction since the rotation is performed in this direction.

3. The line s_2 must form an angle whose measure is equal to $\frac{3}{4}$ of the measure of the angle of rotation with respect to segment OA. In Figure 1.14, the measure of this angle is $\frac{3}{4} \times 120° = 90°$.

After the lines of reflection have been drawn, we need only draw the image of triangle ABC under this series of reflections by applying the method explained in this unit. The image A'B'C' created in this way is equivalent to that obtained under rotation *r*.

It is your turn now! Apply this new method to draw the image of the figure below under the rotation represented. (Recall Practice Exercise 3b) in this unit, where a rotation of 180° was required.)